Tractability of multivariate linear problems for weighted spaces of functions

Woźniakowski H. (Warsaw, Poland)

Multivariate linear problems for spaces of functions of many variables d occur in many applications. Examples of such problems include integration, approximation, and the solution of differential equations. The number d of variables is sometimes in the hundreds or thousands as it is the case for some problems in financial mathematics.

Tractability of multivariate problems has been intensively studied in recent years. This concept is defined in terms of the minimal number $n(\varepsilon, d)$ of function values needed to compute an ε-approximation in the worst case or in the average case setting. *Tractability* means that $n(\varepsilon, d)$ can be bounded by a polynomial in ε^{-1} and d. *Strong tractability* means that $n(\varepsilon, d)$ is independent of d and polynomially dependent on ε^{-1}.

For many classical spaces all variables play the same role, and $n(\varepsilon, d)$ depends exponentially on d. This is called the *curse of dimensionality*, and leads to intractability. The first such an example was given by Bahkvalov in 1959 for multivariate integration of r times continuously differentiable functions. This is also the case for multivariate integration for tensor product Sobolev spaces, and this problem is strictly related to the L_2-discrepancy.

To vanquish the curse of dimensionality, we need to treat variables of functions with diminishing importance. This leads to *weighted* spaces of functions in which the influence of each variable or a group of variables is moderated by the corresponding weight.

In many applications, although d is huge, functions can be well approximated by sums of functions that depend on groups of just a few variables up to a given order k, with the order defined as the number of variables in a group. We can model this situation by *finite-order weights*.

Necessary and sufficient conditions on weights to obtain tractability or strong tractability of linear multivariate problems have been obtained. In particular, for finite-order weights we have tractability or even strong tractability of many multivariate problems. This holds even in the worst case setting for multivariate integration for tensor product Sobolev spaces. Tractability bounds can be achieved by shifted lattice rules with generators computed by the component-by-component algorithm, and by well-known low discrepancy sequences such as Halton, Sobol and Niederreiter sequences. For other multivariate problems such as approximation and the solution of differential equations, tractability bounds are achieved by weighted Smolyak-type algorithms.