On local pseudodimension

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Let F be some set of real-valued functions on S. Let $\mathrm{sgn}(v) = 1$ for $v > 0$, and 0 for $v \leq 0$. We introduce also a function from $\mathbf{R}^m$ to $\{0;1\}^m$ in the following way $\mathrm{sgn}[v] = (\mathrm{sgn}(v_1), \ldots \mathrm{sgn}(v_m))$, $v = (v_1, \ldots v_m)$. We define Vapnik-Cervonnekis dimension $d_{VC}(F)$, as the maximal number $m \in \mathbf{N}$ such, that there exist different points $\tau_1, \ldots \tau_m \in S$ with $\#\{\mathrm{sgn}\,[(f(\tau_1), \ldots f(\tau_m))] : f \in F\} = 2^m$. Let $\dim_P F$ be the greatest $m \in \mathbf{N}$ such, that there exist different points $\tau_1, \ldots \tau_m \in S$ and a vector $v \in \mathbf{R}^m$, with $\#\{\mathrm{sgn}\,[(f(\tau_1), \ldots f(\tau_m)) + v] : f \in F\} = 2^m$. These notions are very useful in mathematical statistics (learning theory) and approximation theory (especially in nonlinear approximations) (see the bibliography in [1],[2],[5]). It seems natural to define "local" pseudodimension for the class of functions F in the following way $\dim_{P,loc} F = \sup_{f \in F} \lim_{\delta \to 0+} d_{VC}\{g - f : g \in F, \|g - f\|_{C(S)} \leq \delta\}$. This quantity can be calculated for many important classes of functions: fewnomials, rational functions, exponential sums, neural networks (see its definition in [5]), generalized splines with free knots (see its definition in [3]). We suppose that S is a segment on $\mathbf{R}$. 1. For the set of exponential sums $E_n := \{\sum_{j=1}^n a_j e^{\alpha_j t} : a_j, \alpha_j \in \mathbf{R}\}$ we have $\dim_{P,loc} E_n = 2n$. 2. For the fewnomials $FP_n = \{f(t) = \sum_{j=1}^n a_j t^{n_j} : a_j \in \mathbf{R}, n_j \in \mathbf{Z}_+\}$ holds $\dim_{P,loc} FP_n = n$, if $S \subset (0; +\infty)$ and $\dim_{P,loc} FP_n = 2n$, if $0 \in \mathrm{int}S$ (the lower estimate was established by S. V. Konyagin). The pseudodimension $\dim_P FP_n$ was estimated in [4]. 3. For the set of algebraic rational functions$R_n = \{r(t) = p(t)/q(t) : p, q \in P_n, r \in C(S)\}$ we have$\dim_{P,loc} R_n = 2n+1$. It is well-known, that $\dim_P F = \dim F$ for any finite-dimensional subspace F in $C(S)$ but (see [3]) pseudodimension of a set can be infinite, while its topological dimension is finite. K. S. Ryutin (in [2]) proved, that for any locally compact subset F in $C(S)$ (S — is a compact metric space) holds the inequality $\dim_P F \geq \dim F$. For a given compact metric space Q, let $D(Q) := \min\{\dim_P(F) : F \subset C[0;1], F \approx Q\}$, where "$\approx$" means homeomorphy. The author established, that: if Q can be embedded (topologically) into some n-dimensional compact simplicial complex (polyhedron), then $D(Q) \leq n$; but for any $n \in \mathbf{N}$ there exists an n-dimensional compact metric space Q_n such, that $D(Q_n) > n$.

References

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