Nonstationary wavelets and one extremal problem for trigonometric polynomials

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Nonstationary wavelets and nonstationary subdivision schemes were introduced in 1990th in the works of I.Novikov, M.Berkolaiko, A.Cohen, N.Dyn and others. In contrast to the usual (stationary) wavelets the corresponding multiresolutional analysis $\{V_j\}_{j\in\mathbb{Z}}$ is constructed by not a single mask m_0, but by a sequence of different masks $\{m_k\}$. For any j the transfer from the space V_j to V_{j+1} is realized by the corresponding mask m_j. Each space V_j is spanned by its own scaling function φ_j. Similarly one defines the nonstationary subdivision schemes. The nonstationarity has made it possible to achieve a better smoothness and a better approximation properties of wavelets/subdivision schemes than those elaborated for the stationary case. In particular, there are infinitely smooth nonstationary wavelets with compact supports, which is impossible in the stationary case. However, in all the corresponding examples the lengths of $\mathrm{supp}\,\varphi_{\mathrm{j}}$ grow faster than 2^j as $j \to \infty$. One of the main problems arise in this context is to find the correlation between the best possible regularity of nonstationary scaling functions φ_j and the growth of their supports. In particular, can they be infinitely smooth if the lengths of the supports grow not faster than $C \cdot 2^j$, as we have in the stationary case? How is the best smoothness restricted by the constant C?

This problem can be solved by considering the following extremal problem: find a sequence $\{m_k\}$ of nonnegative trigonometric polynomials, for which the function

$$f(\xi) = \prod_{k=1}^{\infty} m_k(2^{-k}\xi)$$

has the fastest possible decay as $\xi \to +\infty$. All the polynomials satisfy the following conditions:

$$\deg \mathrm{m_k} \ \le\ \mathrm{C}\,, \quad \|\mathrm{m_k}\| \ \le \mathrm{M}\,, \quad \mathrm{m_k}(0) \ =\ 1$$

(the constants C and M are given). Basically, the estimating of the rate of decay for a given sequence of polynomials is a very difficult problem even for stationary sequences, where all the polynomials m_k coincide. For all pairs C, M we find the fastest decay and characterize all the optimal sequences of polynomials $\{m_k\}$, for which this decay is attained. The answer is given in terms of the asymptotic behavior of zeros of the polynomials. Surprisingly, by weakening the norm boundedness condition one can achieve much faster decay. From this result we derive some conclusions for the nonstationary wavelets and subdivision schemes, and also consider possible generalizations to multivariate polynomials.