Polynomial projectors preserving homogeneous partial differential equations

Dinh Dung (Hanoi, Vietnam)

1. A polynomial projector Π of degree d in $H(\mathbb{C}^n)$ is said to *preserve homogeneous partial differential equations (HPDE)* of degree k if for every $f \in H(\mathbb{C}^n)$ and every homogeneous polynomial of degree k, $q(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$, there holds the implication:

$$q(D)f = 0 \Rightarrow q(D)\Pi(f) = 0.$$

It turns out that a polynomial projector Π preserves HPDE of degree k, $1 \le k \le d$, if and only if there are analytic functionals $\mu_k, \mu_{k+1}, \dots, \mu_d \in H'(\mathbb{C}^n)$ with $\mu_i(1) \neq 0$, $i = k, \dots, d$, such that Π is represented in the following form

$$\Pi(f) = \sum_{|\alpha|<k} a_\alpha(f) u_\alpha + \sum_{k\le|\alpha|\le d} D^\alpha \mu_{|\alpha|} u_\alpha,$$

with some $a_\alpha's \in H'(\mathbb{C}^n)$, $|\alpha| < k$, where $u_\alpha(z) := z^\alpha/\alpha!$. Moreover, there is an example of polynomial projectors preserving HPDE of degree $k > 1$ without preserving HPDE of smaller degree.
(DD, J.-P. Calvi, Nguyen Tien Trung, 2004)

2. A polynomial projector Π can be completely described by the *space of interpolation conditions* $\Im(\Pi) \subset H'(\mathbb{C}^n)$ defined by

$$\Im(\Pi) := \{\, \varphi \in H'(\mathbb{C}^n) : \varphi(f) = \varphi(\Pi(f)) \,\}.$$

For the polynomial projectors that preserve all HPDE, this property is equivalent to the preservation of ridge functions and can be characterized in terms of the space of interpolation conditions. More precisely, the space of interpolation conditions $\Im(\Pi)$ of a polynomial projector Π preserving all HPDE, is and only is of the form

$$\Im(\Pi) = \ \mathrm{span}\{\, D^\alpha \mu_{|\alpha|} : |\alpha| = i,\ i = 0, 1, \dots, d \,\},$$

where $\mu_0, \mu_1, \dots, \mu_d$ are $d+1$ not necessarily distinct analytic functionals from $H'(\mathbb{C}^n)$ such that $\mu_i(1) \neq 0$ for $i = 0, 1, \dots, d$. Several applications are given to the study of the structure of these projectors and, more particularly, to projectors related to Kergin interpolation.
(J.-P. Calvi, L. Filipsson, 2002)

3. Let $n \ge 2$ and Π be a Birkhoff projector of degree d on $\mathbb{C}^n$, i.e., a polynomial projector for which $\Im(\Pi)$ is generated by discrete functionals that is to say by functionals of the form $D^\alpha[a]$. Then Π preserves all HPDE if and only if it is an Abel-Gontcharoff projector, that is, there are $a_0, \dots, a_d \in \mathbb{C}^n$ not necessary distinct such that

$$\Im(\Pi) = \mathrm{span}\{\, D^\alpha[a_s] : |\alpha| = s,\ s = 0, \dots, d \,\}.$$

It is worth noting that this result is typical of the higher dimension. It is indeed not true in dimension 1 in which the concept of projector preserving all HPDE reduces to a triviality.
(DD, J.-P. Calvi, Nguyen Tien Trung, 2004)