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The optimal recovery of a function from an inaccurate information on its *k*-plane transform

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Abstract

We consider the optimal recovery of the βth degree of the Laplacian value on a function from the information on its k-plane transform, measured with an error. We present the error of the optimal recovery and the set of optimal methods on classes with the bounded αth degree of the Laplacian, where $0 \leqslant \beta < \alpha$. As a consequence, we give one inequality for the norms of the degree of the Laplace operator and the k-plane transform. Particular cases include new inversion methods and inequalities for the classical Radon and x-ray transforms.

Keywords: Radon transfrom, optimal recovery, Laplace operator, x-ray transform, k-plane transform, extremal problems, approximations

In general, a problem of the optimal recovery, studied in papers [1–3], is to recover a value of a linear operator on a subset (class) in a linear space from a value of another linear operator (called information), measured with an error in a given metric. In most papers (starting from [4] and recently in [5, 6]) information is considered to be a linear functional or an operator that maps a function to its values on a set of points, its Fourier coefficients or Fourier transform. In the present paper we consider the k-plane transform—an operator that maps a function on $\mathbb{R}^d$ to the set of its integrals over all k-planes. This operator is widely used in computerized tomography theory, which deals with the numerical reconstruction of functions from their linear integrals. Special cases are the Radon transform ($k = d - 1$) and the x-ray transform ($k = 1$). For the particular classes of functions there exist different inversion formulas that allow us to produce an exact reconstruction (see [7]). We consider the case when the k-plane transform is measured with an error $\delta > 0$ in the mean square metric. In the optimal recovery theory operators of this kind previously appeared in [8], where for a function on the unit disk the information is the Radon transform measured in a finite number of directions and paper [9] considering the radial integration operator on the class of analytic functions. In paper [10] we consider the Hardy space h_2 of harmonic functions in a unit ball in $\mathbb{R}^d$, while the information is the radial integration operator, measured with an error. Paper [11] deals with the same class of functions, but under the action of the Radon transform. The solutions in these two problems arise from the structure of the Hardy space and decomposition of functions into a special series by spherical harmonics. The current paper solves a new optimal recovery problem, which broadens the results to a more 'natural' class of functions, which is free from the non-typical to computerized tomography conditions of the Hardy space, and a more general information operator that includes the Radon transform as a particular case. New optimal methods, that present the best approximation of the considered class of functions also bring interest from the point of view of applications as they allow us to perform the numerical reconstruction of a function from inaccurate data. Additionally we exploit the connection between the optimal recovery problem and the inequality for operators that arises from its solution to obtain a new inequality for the k-plane transform.

Consider $G_{k,d}$ the Grassmann manifold of (non-oriented) k-dimensional subspaces in $\mathbb{R}^d$. The orthogonal group $O(d)$ acts transitively on it and for a $\pi \in G_{k,d}$ its stationary subgroup is $O(k) \times O(d-k)$, where $O(k)$ acts on the k-dimensional subspace π and $O(d-k)$—on its orthogonal complement. Thus $G_{k,d}$ can be identified with the homogeneous space $O(d)/(O(k) \times O(d-k))$. By $d\pi$ we denote the $O(d)$-invariant measure on $G_{k,d}$, unique up to a constant. According to [12] the measure of the Grassmanian can be normalized by

$$|G_{k,d}| = \frac{|\mathbb{S}^{d-1}||\mathbb{S}^{d-2}|\ldots|\mathbb{S}^{d-k}|}{2|\mathbb{S}^{k-1}||\mathbb{S}^{k-2}|\ldots|\mathbb{S}^{1}|}, \quad k \geqslant 2,$$
$$|G_{1,d}| = |\mathbb{S}^{d-1}|/2,$$
$$|G_{0,d}| = 1,$$

and the measure of the sphere is

$$|\mathbb{S}^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

From the general theory of Helgason [13] we derive the following formula for the integrable functions on $\mathbb{R}^d$ (corollary 2.4 from [14]):

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}x = \frac{1}{\gamma_{d-k,d}} \int_{G_{k,d}} \int_{\pi^\perp} |x''|^k f(x'')\mathrm{d}x''\mathrm{d}\pi, \tag{1}$$

where $\gamma_{k,d} = |G_{k-1,d-1}|$.

Given the representation of a point $x \in \mathbb{R}^d$ in a form $x = x' + x''$, $x' \in \pi$, $x'' \in \pi^\perp$, the k-plane transform is defined by the integral along the plane parallel to π through the point x'':

$$Pf(\pi, x'') = P_\pi f(x'') = \int_\pi f(x' + x'')\mathrm{d}x', \quad x'' \in \pi^\perp.$$

Its domain is the manifold of all k-planes in $\mathbb{R}^d$ (see figure 1)

$$\mathcal{G}_{k,d} = \{(\pi, x'') : \pi \in G_{k,d}, x'' \in \pi^\perp\}.$$

One important relation between the k-plane transform and the Fourier transform

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\xi} f(x)\mathrm{d}x$$

is known as the projection-slice theorem.

Theorem 1. *If $f \in L_1(\mathbb{R}^d)$, then*

$$\widehat{(P_\pi f)}(\xi'') = (2\pi)^{k/2}\widehat{f}\,(\xi''), \qquad \xi'' \in \pi^\perp.$$

Hilbert space $L_2(\mathcal{G}_{k,d})$ is produced by a scalar product

$$(g, h)_{L_2(\mathcal{G}_{k,d})} = \int_{G_{k,d}} \int_{\pi^\perp} g(\pi, x'')\overline{h(\pi, x'')}\mathrm{d}x''\mathrm{d}\pi.$$

The dual k-plane transform is defined by formula

$$P^\# g(x) = \int_{G_{k,\mathrm{d}}} g(\pi, E_{\pi^\perp}(x))\mathrm{d}\pi,$$

where $E_\pi : \mathbb{R}^d \to \pi$ is the orthogonal projection operator onto $\pi \in G_{k,d}$. Notice that $P^\#$ is the formal adjoint operator of P with duality relation

$$<f, P^\# g>_{L_2(\mathbb{R}^d)} = <Pf, g>_{L_2(\mathcal{G}_{k,d})}. \tag{2}$$

An introduction to that and related formulas can be found in [13–15].

We will work with the class of functions which is constructed through the degree of the Laplace operator, defined for $\alpha > 0$ by the equation

$$\widehat{(-\Delta)^{\alpha/2} f}\,(\xi) = |\xi|^\alpha \widehat{f}\,(\xi)$$

on the set of functions $f \in L_2(\mathbb{R}^d)$ that satisfy the condition $|\xi|^\alpha \widehat{f}\,(\xi) \in L_2(\mathbb{R}^d)$. We will use a shorter notation $\Lambda = (-\Delta)^{1/2}$ and define the class

$$W = \{f \in L_2(\mathbb{R}^d) : \|\Lambda^\alpha f\|_{L_2(\mathbb{R}^d)} \leqslant 1; \quad Pf \in L_2(\mathcal{G}_{k,d})\}.$$

Suppose that for a function Pf we know an approximation $g \in L_2(\mathcal{G}_{k,d})$ such that

$$\|Pf - g\|_{L_2(\mathcal{G}_{k,d})} \leqslant \delta, \qquad \delta > 0.$$

On this information we want to recover function $\Lambda^\beta f$ as an element of $L_2(\mathbb{R}^d)$, where $0 \leqslant \beta < \alpha$. We consider all possible methods or recovery—arbitrary maps $m : L_2(\mathcal{G}_{k,d}) \to L_2(\mathbb{R}^d)$. For every method of recovery m we define its error $e(\delta, m)$ by

$$e(\delta, m) = \sup_{\substack{f \in W, g \in L_2(\mathcal{G}_{k,d}) \\ \|Pf - g\|_{L_2(\mathcal{G}_{k,d})} \leqslant \delta}} ||\Lambda^\beta f - m(g)||_{L_2(\mathbb{R}^d)}.$$

The smallest error among all the methods is called the error of the optimal recovery

$$E(\delta) = \inf_{m: L_2(\mathcal{G}_{k,d}) \to L_2(\mathbb{R}^d)} e(\delta, m). \tag{3}$$

Method m for which the error of the optimal recovery is attained, i.e. $e(\delta, m) = E(\delta)$, is called optimal. Our goal is to present the explicit construction for the optimal methods and the error of the optimal recovery.

When applied to $g(\pi, x'')$ the Fourier transform and operator Λ act on the second variable and we use notation $g_\pi(x'') = g(\pi, x'')$. Define functions $t(\sigma)$, $y(\sigma)$ and constants $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ by formulas

$$t(\sigma) = \frac{\sigma^{2\alpha+k}}{(2\pi)^k \gamma_{d-k,d}}, \qquad y(\sigma) = \frac{\sigma^{2\beta+k}}{(2\pi)^k \gamma_{d-k,d}}, \qquad \sigma \in \mathbb{R}; \tag{4}$$

$$\widehat{\lambda}_1 = ((2\pi)^k \gamma_{d-k,d})^{\frac{2(\beta-\alpha)}{2\alpha+k}} \frac{2\beta + k}{2\alpha + k} \delta^{\frac{4(\alpha-\beta)}{2\alpha+k}}, \quad \widehat{\lambda}_2 = ((2\pi)^k \gamma_{d-k,d})^{\frac{2(\beta-\alpha)}{2\alpha+k}} \frac{2(\alpha - \beta)}{2\alpha + k} \delta^{\frac{-4\beta-2k}{2\alpha+k}}. \tag{5}$$

Theorem 2. *The error of the optimal recovery is given by*

$$E(\delta) = \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2} = ((2\pi)^k \gamma_{d-k,d})^{\frac{\beta-\alpha}{2\alpha+k}} \delta^{\frac{2(\alpha-\beta)}{2\alpha+k}}$$

and the following methods are optimal:

$$m_a(g)(x) = \frac{1}{(2\pi)^k \gamma_{d-k,d}} [P^{\#} \Lambda^k u](x), \tag{6}$$

where

$$\widehat{u}(\xi'') = |\xi''|^\beta a(\xi'') \widehat{g_\pi}(\xi''), \quad \xi'' \in \pi^\perp,$$

$$a(\xi'') = \frac{\widehat{\lambda}_2}{\widehat{\lambda}_1 t(|\xi''|) + \widehat{\lambda}_2} + \varepsilon(\xi'') \frac{\sqrt{\widehat{\lambda}_1 \widehat{\lambda}_2}\, |\xi''|^{\alpha-\beta}}{\widehat{\lambda}_1 t(|\xi''|) + \widehat{\lambda}_2} \sqrt{\widehat{\lambda}_1 t(|\xi''|) + \widehat{\lambda}_2 - y(|\xi''|)}, \tag{7}$$

ε is an arbitrary function satisfying $\|\varepsilon\|_{L_\infty(\mathbb{R}^d)} \leqslant 1$.

Proof. Consider the so-called dual problem to (3):

$$\|\Lambda^\beta f\|^2_{L_2(\mathbb{R}^d)} \to \sup, \quad \|\Lambda^\alpha f\|^2_{L_2(\mathbb{R}^d)} \leqslant 1, \quad \|Pf\|^2_{L_2(\mathcal{G}_{k,d})} \leqslant \delta^2.$$

Its solution gives the lower bound for $E(\delta)$ due to the following inequalities, where m is an arbitrary method:

$$\begin{aligned} e(\delta, m) &= \sup_{\substack{f \in W, g \in L_2(\mathcal{G}_{k,d}) \\ \|Pf - g\|_{L_2(\mathcal{G}_{k,d})} \leqslant \delta}} \|\Lambda^\beta f - m(g)\|_{L_2(\mathbb{R}^d)} \\ &\geqslant \sup_{\substack{f \in W \\ \|Pf\|_{L_2(\mathcal{G}_{k,d})} \leqslant \delta}} \|\Lambda^\beta f - m(0)\|_{L_2(\mathbb{R}^d)} \\ &\geqslant \sup_{\substack{f \in W \\ \|Pf\|_{L_2(\mathcal{G}_{k,d})} \leqslant \delta}} \frac{\|\Lambda^\beta f - m(0)\|_{L_2(\mathbb{R}^d)} + \|-\Lambda^\beta f - m(0)\|_{L_2(\mathbb{R}^d)}}{2} \\ &\geqslant \sup_{\substack{f \in W \\ \|Pf\|_{L_2(\mathcal{G}_{k,d})} \leqslant \delta}} \|\Lambda^\beta f\|_{L_2(\mathbb{R}^d)}. \end{aligned}$$

To obtain the inequalities we notice that if function f is admissible, then function $-f$ is also admissible, i.e. the set W is centrally symmetric. Hence

$$E(\delta) \geqslant \sup_{\substack{f \in W \\ \|Pf\|_{L_2(\mathcal{G}_{k,d})} \leqslant \delta}} \|\Lambda^\beta f\|_{L_2(\mathbb{R}^d)}.$$

We use theorem 1 and equation (1) to transform the functional and the constraints in the dual problem as follows:

$$\|\Lambda^\beta f\|^2_{L_2(\mathbb{R}^d)} = \|\widehat{\Lambda^\beta f}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2\beta} |\widehat{f}(\xi)|^2 \mathrm{d}\xi,$$
$$\|\Lambda^\alpha f\|^2_{L_2(\mathbb{R}^d)} = \|\widehat{\Lambda^\alpha f}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2\alpha} |\widehat{f}(\xi)|^2 \mathrm{d}\xi,$$

$$\|Pf\|^2_{L_2(\mathcal{G}_{k,d})} = \int_{G_{k,d}} \int_{\pi^\perp} |Pf(\pi, x'')|^2 \mathrm{d}x'' \mathrm{d}\pi = \int_{G_{k,d}} \int_{\pi^\perp} |\widehat{(Pf_\pi)}(\eta)|^2 \mathrm{d}\eta \mathrm{d}\pi$$
$$= (2\pi)^k \int_{G_{k,d}} \int_{\pi^\perp} |\widehat{f}(\eta)|^2 \mathrm{d}\eta \mathrm{d}\pi = (2\pi)^k \gamma_{d-k,d} \int_{\mathbb{R}^d} \frac{1}{|\xi|^k} |\widehat{f}(\xi)|^2 \mathrm{d}\xi.$$

If we denote $|\widehat{f}(\xi)|^2 \mathrm{d}\xi = \mathrm{d}\mu(\xi)$ the dual problem can be presented as

$$\int_{\mathbb{R}^d} |\xi|^{2\beta} \mathrm{d}\mu \to \sup, \quad \int_{\mathbb{R}^d} |\xi|^{2\alpha} \mathrm{d}\mu \leqslant 1, \quad \int_{\mathbb{R}^d} \frac{(2\pi)^k \gamma_{d-k,d}}{|\xi|^k} \mathrm{d}\mu \leqslant \delta^2. \tag{8}$$

Now we consider (8) to be a new extremal problem, where $\mathrm{d}\mu(\xi)$ is an arbitrary measure. Obviously its solution is not less than the solution of the original dual problem. To solve the dual problem we will present the solution of (8) and the sequence of admissible functions that bring the same value in the dual problem. Consider the Lagrange function of (8):

$$L(\mathrm{d}\mu, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2 \delta^2$$
$$+ (2\pi)^k \gamma_{d-k,d} \int_{\mathbb{R}^d} \frac{1}{|\xi|^k} \left(\lambda_1 \frac{|\xi|^{2\alpha+k}}{(2\pi)^k \gamma_{d-k,d}} + \lambda_2 - \frac{|\xi|^{2\beta+k}}{(2\pi)^k \gamma_{d-k,d}} \right) \mathrm{d}\mu$$

or, using notations (4):

$$L(\mathrm{d}\mu, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2 \delta^2 + (2\pi)^k \gamma_{d-k,d} \int_{\mathbb{R}^d} \frac{1}{|\xi|^k} (\lambda_1 t(|\xi|) + \lambda_2 - y(|\xi|)) \mathrm{d}\mu.$$

If there exist the Lagrange multipliers $\widehat{\lambda}_1, \widehat{\lambda}_2 \geqslant 0$ and the measure $\mathrm{d}\mu^*$, which are admissible in (8), that minimize the Lagrange function, i.e.

$$\min_{\mathrm{d}\mu \geqslant 0} L(\mathrm{d}\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = L(\mathrm{d}\mu^*, \widehat{\lambda}_1, \widehat{\lambda}_2),$$

and satisfy

$$\widehat{\lambda}_1 \left(\int_{\mathbb{R}^d} |\xi|^{2\alpha} \mathrm{d}\mu^* - 1 \right) + \widehat{\lambda}_2 \left((2\pi)^k \gamma_{d-k,d} \int_{\mathbb{R}^d} \frac{\mathrm{d}\mu^*}{|\xi|^k} - \delta^2 \right) = 0$$

(complementary slackness condition), then $\mathrm{d}\mu^*$ brings maximum to (8). We shall present such $\widehat{\lambda}_1, \widehat{\lambda}_2$ and $\mathrm{d}\mu^*$. Consider a function given parametrically by equations (4) or explicitly

$$y(t) = ((2\pi)^k \gamma_{d-k,d})^{\frac{2\beta-2\alpha}{2\alpha+k}} t^{\frac{2\beta+k}{2\alpha+k}}, \quad t \geqslant 0.$$

It's concave for $0 \leqslant \beta < \alpha$. The equation of the tangent line to $y(t)$ at a point $1/\delta^2$ (the corresponding value of σ is $\sigma^* = [(2\pi)^k \gamma_{d-k,d} \delta^{-2}]^{1/(2\alpha+k)}$) is $u = \widehat{\lambda}_1 t + \widehat{\lambda}_2$, where $\widehat{\lambda}_1, \widehat{\lambda}_2$ defined in (5). Thus, we have $\widehat{\lambda}_1 t(\sigma) + \widehat{\lambda}_2 - y(\sigma) \geqslant 0$ (see figure 2) and $L(\mathrm{d}\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) \geqslant -\widehat{\lambda}_1 - \widehat{\lambda}_2 \delta^2$. Consider a measure supported on the sphere $|\xi| = \sigma^*$ (i.e. the surface δ-function)

$$\mathrm{d}\mu^* = \frac{(\sigma^*)^{-d+1-2\alpha}}{|\mathbb{S}^{d-1}|} \delta_{|\xi|=\sigma^*}.$$

This is admissible in (8), satisfies the complementary slackness condition and minimizes the Lagrange function, as $L(\mathrm{d}\mu^*, \widehat{\lambda}_1, \widehat{\lambda}_2) = -\widehat{\lambda}_1 - \widehat{\lambda}_2 \delta^2$. Thus, it brings the extremum in problem (8), whose solution is equal to $\widehat{\lambda}_1 + \widehat{\lambda}_2\delta^2$. By a standard approximation of the δ-function it is easy to show that the solution of the dual problem is the same as in (8). Thereby we obtain inequality $E(\delta) \geqslant \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2\delta^2}$, which represents a lower bound for the error of the optimal recovery.

Now we show that the error of the methods (6) is equal to the achieved estimate. First we notice an isometry property of the k-plane transform (theorem 3.97 from [15])

$$\|\Lambda^{k/2}Pf\|^2_{L_2(\mathcal{G}_{k,d})} = (2\pi)^k\gamma_{d-k,d}\,\|f\|^2_{L_2(\mathbb{R}^d)},$$

which is obtained from the Plancherel's theorem, theorem 1 and formula (1) by

$$\begin{aligned}\|\Lambda^{k/2}Pf\|^2_{L_2(\mathcal{G}_{k,d})} &= \int_{G_{k,d}}\int_{\pi^\perp}|\Lambda^{k/2}Pf(\pi, x'')|^2\mathrm{d}x''\mathrm{d}\pi\\ &= \int_{G_{k,d}}\int_{\pi^\perp}|\widehat{\Lambda^{k/2}Pf}(\pi, \xi'')|^2\mathrm{d}\xi''\mathrm{d}\pi = \int_{G_{k,d}}\int_{\pi^\perp}(2\pi)^k|\xi''|^k\,|\widehat{f}(\xi'')|^2\mathrm{d}\xi''\mathrm{d}\pi\\ &= (2\pi)^k\gamma_{d-k,d}\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2\mathrm{d}\xi = (2\pi)^k\gamma_{d-k,d}\,\|f\|^2_{L_2(\mathbb{R}^d)}.\end{aligned}$$

From this property and duality equation (2) it follows that

$$\begin{aligned}\|P^\#\Lambda^k g\|^2_{L_2(\mathbb{R}^d)} &= |< P^\#\Lambda^k g, P^\#\Lambda^k g>_{L_2(\mathbb{R}^d)}| = |<PP^\#\Lambda^k g, \Lambda^k g>_{L_2(\mathcal{G}_{k,d})}|\\ &= |<\Lambda^{k/2}PP^\#\Lambda^k g, \Lambda^{k/2}g>_{L_2(\mathcal{G}_{k,d})}| \leqslant \|\Lambda^{k/2}PP^\#\Lambda^k g\|_{L_2(\mathcal{G}_{k,d})}\,\|\Lambda^{k/2}g\|_{L_2(\mathcal{G}_{k,d})}\\ &= \sqrt{(2\pi)^k\gamma_{d-k,d}}\,\|P^\#\Lambda^k g\|_{L_2(\mathbb{R}^d)}\,\|\Lambda^{k/2}g\|_{L_2(G_{k,d})}\end{aligned}$$

which results in the inequality

$$\|P^\#\Lambda^k g\|_{L_2(\mathbb{R}^d)} \leqslant \sqrt{(2\pi)^k\gamma_{d-k,d}}\,\|\Lambda^{k/2}g\|_{L_2(G_{k,d})}. \tag{9}$$

By taking the inverse Fourier transform from both parts in the projection-slice theorem 1 we gain the following representation for function f

$$\begin{aligned}f(x) &= (2\pi)^{-d/2}\int_{G_{k,d}}\int_{\pi^\perp}\frac{|\xi''|^k}{\gamma_{d-k,d}}(2\pi)^{-k/2}\widehat{P_\pi f}(\xi'')\mathrm{e}^{<x,\xi''>}\mathrm{d}\xi''\mathrm{d}\pi\\ &= \frac{(2\pi)^{(-d-k)/2}}{\gamma_{d-k,d}}\int_{G_{k,d}}\int_{\pi^\perp}|\xi''|^k\widehat{P_\pi f}(\xi'')\mathrm{e}^{<Proj_{\pi^\perp}x,\xi''>}\mathrm{d}\xi''\mathrm{d}\pi\\ &= \frac{(2\pi)^{(-d-k)/2}}{\gamma_{d-k,d}}\int_{G_{k,d}}\int_{\pi^\perp}\widehat{\Lambda^k P_\pi f}(\xi'')\mathrm{e}^{<Proj_{\pi^\perp}x,\xi''>}\mathrm{d}\xi''\mathrm{d}\pi\\ &= \frac{(2\pi)^{(-d-k)/2}}{\gamma_{d-k,d}}(2\pi^{(d-k/2)})\int_{G_{k,d}}\Lambda^k Pf(Proj_{\pi^\perp}x)\mathrm{d}\pi = \frac{1}{(2\pi)^k\gamma_{d-k,d}}P^\#\Lambda^k Pf(x).\end{aligned}$$

We use this observation and inequality (9) to obtain an upper bound for the error of the methods (6).

$$
\begin{aligned}
\|\Lambda^{\beta}f - m_a(g)\|^2 &= \left\| \frac{1}{(2\pi)^k \gamma_{d-k,d}} P^{\#}\Lambda^k (P\Lambda^{\beta} f - u) \right\|^2 \\
&\leqslant \frac{1}{(2\pi)^k \gamma_{d-k,d}} \ \|\Lambda^{k/2}(P\Lambda^{\beta} f - u)\|^2 \\
&= \frac{1}{(2\pi)^k \gamma_{d-k,d}} \int_{G_{k,d}} \int_{\pi^{\perp}} |\xi''|^k \, |\widehat{P\Lambda^{\beta} f}(\xi'') - |\xi''|^{\beta} a(\xi'') \widehat{g}_{\pi}(\xi'')|^2 \mathrm{d}\xi'' \mathrm{d}\pi \\
&= \int_{G_{k,d}} \int_{\pi^{\perp}} \frac{|\xi''|^k}{\gamma_{d-k,d}} \ ||\xi''|^{\beta} \widehat{f}\,(\xi'') - (2\pi)^{-k/2} |\xi''|^{\beta} a(\xi'') \widehat{g}_{\pi}(\xi'')|^2 \mathrm{d}\xi'' \mathrm{d}\pi \\
&= \int_{G_{k,d}} \int_{\pi^{\perp}} \frac{|\xi''|^k}{\gamma_{d-k,d}} \ ||\xi''|^{\beta} a(\xi'') (2\pi)^{-k/2} (\widehat{g}_{\pi}(\xi'') - (2\pi)^{k/2} \widehat{f}\,(\xi'')) \\
&\quad + \widehat{f}\,(\xi'') |\xi''|^{\beta} (a(\xi'') - 1)|^2 \ \mathrm{d}\xi'' \mathrm{d}\pi.
\end{aligned}
$$

We transform this expression by applying the Cauchy–Schwarz inequality $|qz| \leqslant |z||q|$ to vectors

$$
z = \left((2\pi)^{-k/2} \frac{|\xi''|^{\beta} a(\xi'')}{\sqrt{\widehat{\lambda}_2}}, \ \frac{\sqrt{\gamma_{d-k,d}}}{|\xi''|^{\frac{k+2\alpha}{2}}} \frac{|\xi''|^{\beta}(a(\xi'') - 1)}{\sqrt{\widehat{\lambda}_1}} \right),
$$

$$
q = \left((\widehat{g}_{\pi}(\xi'') - (2\pi)^{k/2} \widehat{f}\,(\xi''))\sqrt{\widehat{\lambda}_2}, \ \frac{|\xi''|^{\frac{k+2\alpha}{2}}}{\sqrt{\gamma_{d-k,d}}} \sqrt{\widehat{\lambda}_1} \, \widehat{f}\,(\xi'') \right)
$$

to obtain

$$
\begin{aligned}
&\|\Lambda^{\beta} f - m_a(g)\|^2_{L_2(\mathbb{R}^d)} \\
&\leqslant \int_{G_{k,d}} \int_{\pi^{\perp}} A(\xi'') \left(\frac{|\xi''|^{k+2\alpha}}{\gamma_{d-k,d}} \widehat{\lambda}_1 \, |\widehat{f}\,(\xi'')|^2 + |\widehat{g}_{\pi}(\xi'') - (2\pi)^{k/2} \widehat{f}\,(\xi'')|^2 \, \widehat{\lambda}_2 \right) \mathrm{d}\xi'' \mathrm{d}\pi,
\end{aligned}
$$

where

$$
A(\xi'') = \frac{|\xi''|^{k+2\beta}}{\gamma_{d-k,d}} \left((2\pi)^{-k} \frac{a^2(\xi'')}{\widehat{\lambda}_2} + \frac{\gamma_{d-k,d}}{|\xi''|^{k+2\alpha}} \frac{(a(\xi'') - 1)^2}{\widehat{\lambda}_1} \right).
$$

Here $A(\xi'') \leqslant 1$ due to (7) and other terms are estimated by the constraints of the class W to achieve $\|\Lambda^{\beta} f - m_a(g)\|^2_{L_2(\mathbb{R}^d)} \leqslant \widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2$, which ends the proof. □

The design of the optimal methods actually applies a filter $a(\xi'')$ to measurements and instead of the k-plane transform we deal with its Fourier image. This filter defines the amount of information that we use for the optimal recovery. When $a(\xi'')$ can be chosen equal to 1 the corresponding volume of information does not need to be filtered. On the other hand some information is unnecessary as it may not be used by the optimal method, when $a(\xi'')$ can be equal to zero (see figure 3). The following corollary shows that for sufficiently small $|\xi''|$ information $\widehat{g}_{\pi}(\xi'')$ does not need to be filtered and, in contrast, for large $|\xi''|$ the information is useless, as it has no effect on the error of the optimal recovery.

Corollary 1. *In the conditions of theorem 2 the following methods are optimal*

$$m_a(g)(x) = \frac{1}{(2\pi)^k \gamma_{d-k,d}} [P^{\#} \Lambda^k u](x),$$

where

$$\hat{u}(\xi'') = |\xi''|^{\beta} a(\xi'') \widehat{g_{\pi}}(\xi''), \qquad \xi'' \in \pi^{\perp},$$

$$a(\xi'') = \begin{cases} 1 & |\xi''| \leqslant \tau_1, \\ \frac{\hat{\lambda}_2}{\hat{\lambda}_1 t(|\xi''|) + \hat{\lambda}_2} + \varepsilon(\xi'') \frac{\sqrt{\hat{\lambda}_1 \hat{\lambda}_2}\, |\xi''|^{\alpha-\beta}}{\hat{\lambda}_1 t(|\xi''|) + \hat{\lambda}_2} \sqrt{t(|\xi''|)\hat{\lambda}_1 + \hat{\lambda}_2 - y(|\xi''|)} & \tau_1 \leqslant |\xi''| \leqslant \tau_2, \\ 0 & |\xi''| \geqslant \tau_2, \end{cases}$$

ε *is an arbitrary function satisfying* $\|\varepsilon\|_{L_\infty(\mathbb{R}^d)} \leqslant 1$, $\tau_1 = ((2\pi)^k \hat{\lambda}_2 \gamma_{d-k,d})^{\frac{1}{k+2\beta}}$, $\tau_2 = \hat{\lambda}_1^{\frac{-1}{2(\alpha-\beta)}}$.

Proof. As we have seen in the proof of the theorem 2 the condition on $a(\xi'')$ for the method $m_a(g)$ to be optimal is $A(\xi'') \leqslant 1$. Put $a(\xi'') = 1$ to this inequality and solve it for ξ'' to obtain $|\xi''| \leqslant ((2\pi)^k \hat{\lambda}_2 \gamma_{d-k,d})^{\frac{1}{k+2\beta}}$. Similarly put $a(\xi'') = 0$, then $A(\xi'') \leqslant 1$ is true when $|\xi''| \geqslant \hat{\lambda}_1^{\frac{-1}{2(\alpha-\beta)}}$. □

Another application of theorem 2 is a new inequality for the norms of the k-plane transform and the degree of the Laplace operator.

Corollary 2. *The following exact inequality takes place for a function* $f \in L_2(\mathbb{R}^d)$ *such that* $|\xi|^{\beta} \hat{f}(\xi) \in L_2(\mathbb{R}^d)$, $|\xi|^{\alpha} \hat{f}(\xi) \in L_2(\mathbb{R}^d)$, $Pf \in L_2(\mathcal{G}_{k,d})$, $0 \leqslant \beta < \alpha$:

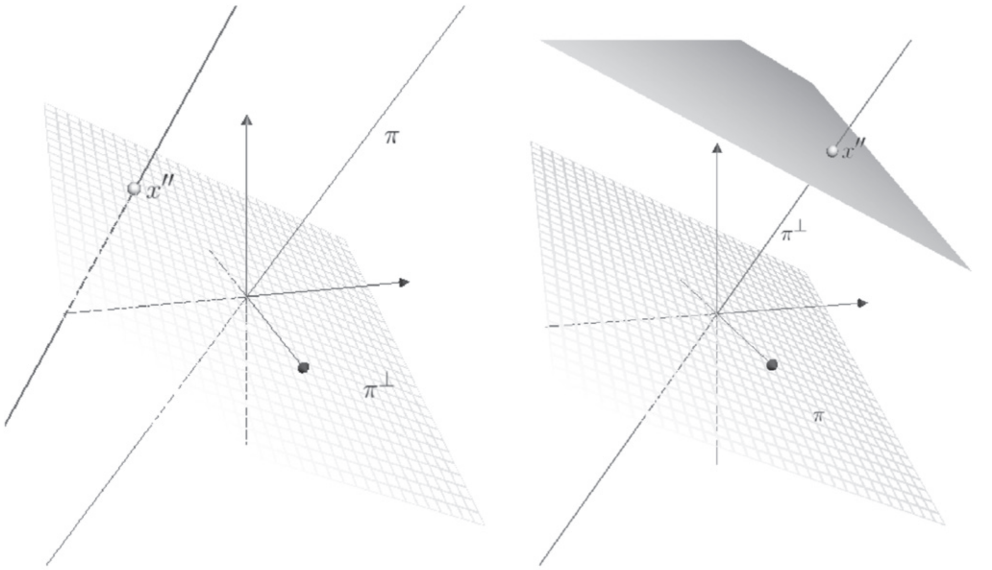
$$\|\Lambda^{\beta} f\|_{L_2(\mathbb{R}^d)} \leqslant ((2\pi)^k \gamma_{d-k,d})^{\frac{\beta-\alpha}{2\alpha+k}} \|Pf\|_{L_2(\mathcal{G}_{k,d})}^{\frac{2(\alpha-\beta)}{2\alpha+k}} \|\Lambda^{\alpha} f\|_{L_2(\mathbb{R}^d)}^{\frac{k-2\beta}{2\alpha+k}}.$$

Proof. From the solution of the dual problem in theorem 2 it follows, that $\|\Lambda^{\beta} v\|_{L_2(\mathbb{R}^d)} \leqslant E(\delta) = ((2\pi)^k \gamma_{d-k,d})^{\frac{\beta-\alpha}{2\alpha+k}} \delta^{\frac{2(\alpha-\beta)}{2\alpha+k}}$, when the following constraints are satisfied: $\|Pv\|_{L_2(\mathcal{G}_{k,d})} = \delta$ and $\|\Lambda^{\alpha} v\|_{L_2(\mathbb{R}^d)} = 1$. So the expression can be presented as $\|\Lambda^{\beta} v\|_{L_2(\mathbb{R}^d)} \leqslant ((2\pi)^k \gamma_{d-k,d})^{\frac{\beta-\alpha}{2\alpha+k}} \|Pv\|_{L_2(\mathcal{G}_{k,d})}^{\frac{2(\alpha-\beta)}{2\alpha+k}}$. Now we put $v(x) = \frac{f(x)}{\|\Lambda^{\alpha} f\|_{L_2(\mathbb{R}^d)}}$, $f \neq 0$ to obtain

$$\|\Lambda^{\beta} f\|_{L_2(\mathbb{R}^d)} \leqslant ((2\pi)^k \gamma_{d-k,d})^{\frac{\beta-\alpha}{2\alpha+k}} \|Pf\|_{L_2(\mathcal{G}_{k,d})}^{\frac{2(\alpha-\beta)}{2\alpha+k}} \|\Lambda^{\alpha} f\|_{L_2(\mathbb{R}^d)}^{\frac{k-2\beta}{2\alpha+k}}.$$

□

As we have already mentioned some particular cases of the presented problem bring the most interest. In computerized tomography theory the general problem is to recover a function itself from different tomographic data, which corresponds to $\beta = 0$ in our notations. A special case of $\beta = 1$ is studied in the local tomography theory, where one of the methods is the so-called Lambda tomography. Instead of function f itself it deals with the related function $Lf = \Lambda f + \mu \Lambda^{-1} f$. This has the advantage that the reconstruction is strictly local in the sense that computation of $Lf(x)$ requires only integrals over lines passing arbitrarily close to x. The details can be found in [16]. The results for the x-ray transform totally correspond to the theorem 2, corollaries 1 and 2 by putting $\beta = 0$ and $k = 1$. The case of the Radon transform needs an additional remark as its usual definition differs from the one we use here. Let



(a) A line parallel to π goes through point x'' in plane $\pi^\perp$

(b) A plane parallel to π goes through point x'' in line $\pi^\perp$

Figure 1. Parametrization of the domains for x-ray ($k = 1$) and Radon ($k = 2$) transforms in $\mathbb{R}^3$.

$s \in \mathbb{R}$, $\theta \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d$, then the Radon transform is defined on $Z = \mathbb{R} \times \mathbb{S}^{d-1}$ by the formula

$$Rf(\theta, s) = \int_{x\theta = s} f(x)\mathrm{d}x.$$

Clearly function $Rf(\theta, s)$ has the same value as $Pf(\pi, x'')$, where $\pi = \{x \in \mathbb{R}^d | x\theta = 0\}$, $x'' = s\theta$ as well as

$$\|Rf\|_{L_2(Z)} = \sqrt{2}\,\|Pf\|_{L_2(\mathcal{G}_{k,d})}, \quad k = d - 1. \tag{10}$$

Therefore class W for $\beta = 0$ can be equivalently presented as

$$W = \{f \in L_2(\mathbb{R}^d) : \|\Lambda^\alpha f\|_{L_2(\mathbb{R}^d)} \leqslant 1; \quad Rf \in L_2(Z)\}, \quad \alpha > 0,$$

and the error of the optimal recovery has the form

$$E(\delta) = \inf_{m: L_2(Z) \to L_2(\mathbb{R}^d)} \sup_{\substack{f \in W, g \in L_2(Z) \\ \|Rf - g\|_{L_2(Z)} \leqslant \delta}} \|f - m(g)\|_{L_2(\mathbb{R}^d)}.$$

From (10) it follows that the solution of this problem is equivalent to the solution in theorem 2 where δ is substituted by $\delta/\sqrt{2}$ in the expressions for λ_1, λ_2 and $E(\delta)$:

$$\widehat{\lambda}_1 = (2\pi)^{\frac{2\alpha(1-d)}{2\alpha+d-1}} \frac{d-1}{2\alpha + d - 1} \left(\frac{\delta}{\sqrt{2}}\right)^{\frac{4\alpha}{2\alpha+d-1}}, \quad \widehat{\lambda}_2 = (2\pi)^{\frac{2\alpha(1-d)}{2\alpha+d-1}} \frac{2\alpha}{2\alpha + d - 1} \left(\frac{\delta}{\sqrt{2}}\right)^{\frac{2(1-d)}{2\alpha+d-1}},$$

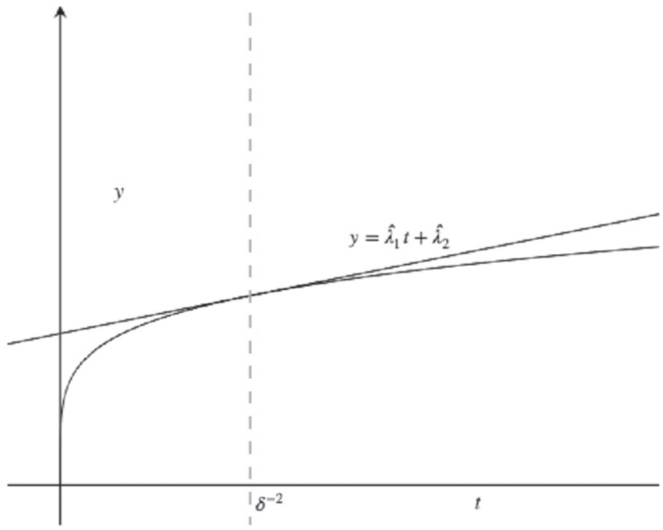


Figure 2. The figure shows function $y(t)$ and the corresponding tangent line for $d = 2$, $k = 1$, $\beta = 0$, $\alpha = 2$, $\delta = 1$.

$$E(\delta) = \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2 \frac{\delta^2}{2}} = (2\pi)^{\frac{\alpha(1-d)}{2\alpha+d-1}} \left(\frac{\delta}{\sqrt{2}}\right)^{\frac{2\alpha}{2\alpha+d-1}}.$$

To present the optimal methods we denote by $g^*(\pi, x'')$ the function on $\mathcal{G}_{d-1,d}$ that corresponds to $g(\theta, s)$ and we notice that $\widehat{g_\pi^*}(\xi'') = \widehat{g_\theta}(\sigma)$, where $\xi'' = \sigma\theta$. Then substitute $g^*(\pi, x'')$ into (6) to obtain

$$m_a(g)(x) = (2\pi)^{1-d}[P^\# \Lambda^{d-1} u](x), \quad \hat{u}(\sigma\theta) = a(\sigma)\widehat{g_\theta}(\sigma), \quad \sigma \in [0, \infty), \quad \theta \in \mathbb{S}^{d-1},$$

where $g_\theta(s) = g(\theta, s)$ and function a can be presented according to the corollary 1 as

$$a(\sigma) = \begin{cases} 1 & 0 \leqslant \sigma \leqslant (2\pi)\widehat{\lambda}_2^{\frac{1}{d-1}}, \\ \frac{\widehat{\lambda}_2}{\widehat{\lambda}_1 t(\sigma) + \widehat{\lambda}_2} + \varepsilon(\sigma) \frac{\sqrt{\widehat{\lambda}_1 \widehat{\lambda}_2}\sigma^\alpha}{\widehat{\lambda}_1 t(\sigma) + \widehat{\lambda}_2} \sqrt{t(\sigma)\widehat{\lambda}_1 + \widehat{\lambda}_2 - y(\sigma)}, & (2\pi)\widehat{\lambda}_2^{\frac{1}{d-1}} \leqslant \sigma \leqslant \widehat{\lambda}_1^{\frac{-1}{2\alpha}}, \\ 0 & \sigma \geqslant \widehat{\lambda}_1^{\frac{-1}{2\alpha}}, \end{cases}$$

ε is an arbitrary function satisfying $\|\varepsilon\|_{L_\infty(\mathbb{R})} \leqslant 1$. Finally, following corollary 2 the Radon transform satisfies the inequality

$$\|f\|_{L_2(\mathbb{R}^d)} \leqslant (2\pi)^{\frac{\alpha(1-d)}{2\alpha+d-1}} 2^{\frac{-\alpha}{2\alpha+d-1}} \|Rf\|_{L_2(Z)}^{\frac{2\alpha}{2\alpha+d-1}} \|\Lambda^\alpha f\|_{L_2(\mathbb{R}^d)}^{\frac{d-1}{2\alpha+d-1}}, \quad \alpha > 0.$$

To visualize the results we bring two numerical examples. In the first example we take the Gaussian function $f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{-|x|^2}{2}}$, which belongs to class W for $d = 2$, $k = 1$, $\alpha = 2$. Its Radon transform is independent of θ (also a Gaussian), that we denote $Rf(s)$. We add error in the form

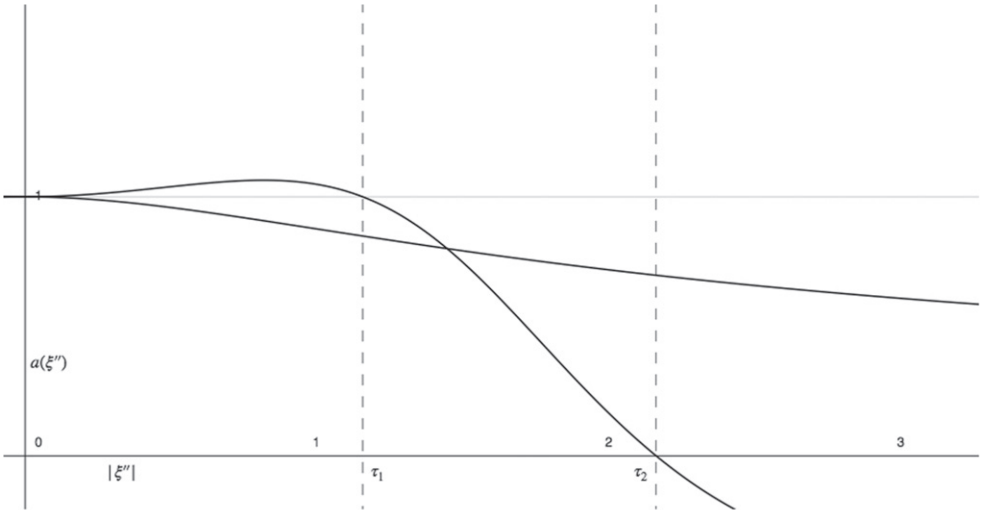


Figure 3. The optimal values of filter a lie between two graphs: one represents function a from theorem 2 for $\epsilon(\xi'') = 1$, another for $\epsilon(\xi'') = -1$. Here $d = 2$, $k = 1$, $\beta = 0$, $\alpha = 2$, $\delta = 1$.

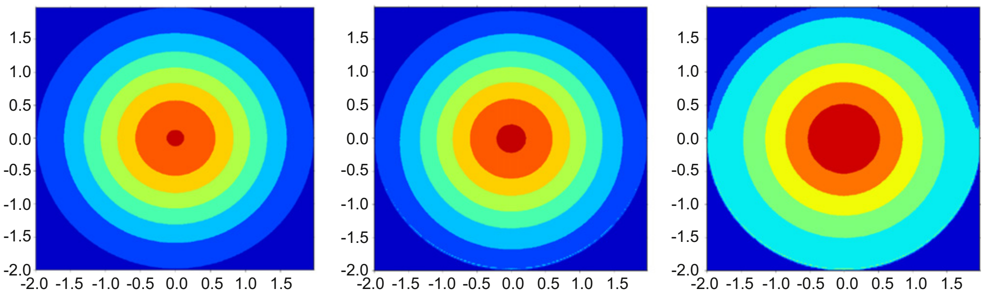


Figure 4. Contour plot of the original image (left), its reconstruction from the inaccurate Radon transform by the optimal method (middle) and 'natural' method (right).

$$e(s) = \begin{cases} a \ * \ \sin(s) + b \ * \ \cos(s), & s \in [-1, 1] \\ 0 & s \notin [-1, 1] \end{cases}$$

where a and b are random numbers. The inaccurate approximation of the Radon transform in this case is presented by $g(s) = Rf(s) + \delta \cdot e(s)/\|e\|_{L_2(Z)}$, where $\delta =$ noise level $*$ $\|Rf\|_{L_\infty}$. We apply the optimal method (we choose $\epsilon(\sigma) = 1$) to recover function f for a noise level of 0.1 and compare it to the recovery by the 'natural' method, which assumes $a(\sigma) = 1$ and actually presents the standard filtered backprojection algorithm (FBP).

The results of the recovery are presented in figures 4 and 6. Images show, that the optimal method provides a more accurate result with a smaller error than the 'natural' method. For the next example we use a standard Shepp-Logan phantom smoothed with Gaussian kernel, as presented on figure 5. As previously we approximate the Radon transform with $g(\theta, s) = Rf(\theta, s) + \delta \cdot e(\theta, s)/\|e\|_{L_2(Z)}$, where $e(\theta, s)$ is a Gaussian distributed noise with zero mean and standard deviation 1 and $\delta =$ noise level $* \|Rf\|_{L_\infty(Z)}$. We apply the optimal

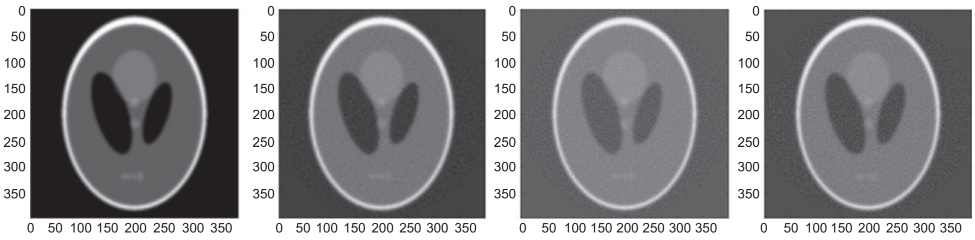


Figure 5. From left to right: original phantom, reconstruction by the optimal method, reconstruction by FBP (Ram-Lak filter), reconstruction by FBP with a Hamming filter.

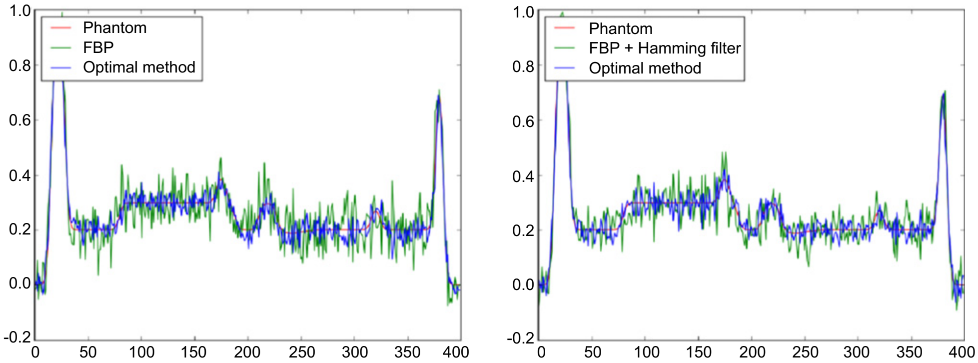


Figure 6. The slice of the original function, its reconstruction by FBP (Ram-Lak), FBP with a Hamming filter and the optimal method.

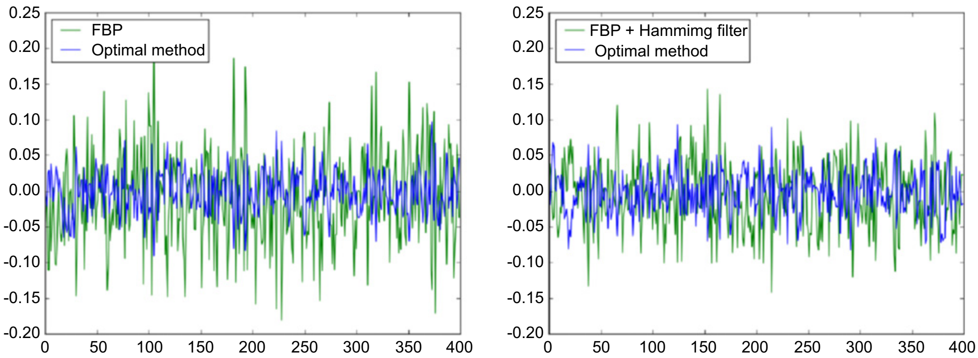


Figure 7. The error of the reconstruction (on the same slice as on figure 6) by FBP (Ram-Lak), FBP with a Hamming filter and the optimal method.

method ($\epsilon(\sigma) = 1$) on a noise level of 0.05 and compare it to the recovery by standard FBP algorithm (Ram-Lack filter) and FBP with Hamming filter. The original image and its reconstructions are presented on figure 5. Figure 6 shows the slice of the original and reconstructed images; figure 7 illustrates the error on the corresponding slice.

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