# OPTIMAL RECOVERY OF FUNCTIONS AND SOLUTIONS OF EVOLUTIONARY EQUATIONS

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We begin with one problem which was a stimulus for the development of general optimal recovery problem. Assume that for a sufficiently smooth function $x(\cdot)$ we know finite number of Fourier coefficients which are given with some accuracy. How to reconstruct the function $x(\cdot)$ or its derivative?

Suppose that

$$x(t)=\sum_{j=-\infty}^{+\infty} x_j e^{ijt}$$

and we know $y_j$, $|j|\le N$, such that

$$\|x^N-y^N\|_{l_2^N}\le\delta,$$

where

$$x^N=\{x_j\}_{|j|\le N},\quad y^N=\{y_j\}_{|j|\le N},$$

and $\|\cdot\|_{l_2^N}$ is the standard Euclidean norm in $\mathbb{R}^N$. The question is how to recover the $k$-th derivative of $x(\cdot)$ (for example, $x'(\cdot)$) knowing the vector $y^N$.

One of the simplest algorithm is the following

$$x'(t)\approx\sum_{|j|\le N} ijy_je^{ijt}.$$

But it is not good since for large $j$ the error of the term $ijy_je^{ijt}$ becomes large. In practice this phenomena is well known. Those who deal with such problems simply cut the terms with high frequencies or smooth them by some filter.

The problem which we would like to pose is: what is a best method of recovery (or, in other words, what is a best filter)?

For this problem it is possible to obtain some algorithms using Tikhonov regularization. However the estimates of such methods are obtained for $\delta$ tends to 0. And we want to obtain a good algorithm for a fixed $\delta$. Moreover, we want to compare various methods and choose the best one in some sense. The idea of searching the best algorithm

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is coming from A.N. Kolmogorov (it was appeared in his papers devoted to $n$-widths). In the simplest form this idea may be seen in the problems connected with best quadrature formulae.

Now let as consider the precise statement of the problem. To avoid technical details it is convenient to deal not with periodic functions and their Fourier coefficients but with functions defined on $\mathbb{R}$ and their Fourier transforms (nevertheless all considered results may be also formulated for periodic case).

Assume that $x(\cdot)$ is sufficiently smooth function defined on $\mathbb{R}$ and we know its Fourier transform $Fx(\cdot)$ on the interval $\Delta_\sigma = (-\sigma, \sigma)$, $0 < \sigma \le \infty$, with some accuracy. More precisely, we know a function $y(\cdot) \in L_2(\Delta_\sigma)$ such that

$$\|Fx(\cdot) - y(\cdot)\|_{L_2(\Delta_\sigma)} \le \delta.$$

We want to recover $x^{(k)}(\cdot)$ knowing $y(\cdot)$.

To make the statement of the problem correct we should know some additional a priory information about function $x(\cdot)$. Usually this information is giving as a class of functions to which $x(\cdot)$ belongs. Thus we assume that in this problem we deal only with functions from the given class. We will consider Sobolev classes of functions.

Set

$$\mathcal{W}_2^r(\mathbb{R}) = \{\, x(\cdot) \in L_2(\mathbb{R}) : x^{(r-1)} \text{ loc. abs. cont.},\ x^{(r)}(\cdot) \in L_2(\mathbb{R}) \,\},$$

where $r$ is a natural number,

$$W_2^r(\mathbb{R}) = \{\, x(\cdot) \in \mathcal{W}_2^r(\mathbb{R}) : \|x^{(r)}(\cdot)\|_{L_2(\mathbb{R})} \le 1 \,\}.$$

Let $x(\cdot) \in W_2^r(\mathbb{R})$ and we know a function $y(\cdot)$ such that

$$\|Fx(\cdot) - y(\cdot)\|_{L_2(\Delta_\sigma)} \le \delta.$$

Any mapping

$$m\colon L_2(\Delta_\sigma) \to L_2(\mathbb{R})$$

we consider as a method (or algorithm) of recovery of $x^{(k)}(\cdot)$, $0 \le k \le r-1$. Note that we do not require any additional properties of $m$ (such as linearity or continuality).

The error of the method $m$ is defined as follows

$$e_\sigma(D^k, W_2^r(\mathbb{R}), \delta, m) = \sup_{\substack{x(\cdot)\in W_2^r(\mathbb{R}),\ y(\cdot)\in L_2(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_2(\Delta_\sigma)}\le\delta}} \|x^{(k)}(\cdot) - m(y)(\cdot)\|_{L_2(\mathbb{R})}.$$

The error of optimal recovery is the following value

$$E_\sigma(D^k, W_2^r(\mathbb{R}), \delta) = \inf_{m\colon L_2(\Delta_\sigma)\to L_2(\mathbb{R})} e_\sigma(D^k, W_2^r(\mathbb{R}), \delta, m).$$

We call $\widehat{m}$ an optimal method of recovery if

$$e_\sigma(D^k, W_2^r(\mathbb{R}), \delta, \widehat{m}) = E_\sigma(D^k, W_2^r(\mathbb{R}), \delta). \tag{1}$$

The solution of the considered problem is given in the following theorem obtained in [1]

**Theorem 1.** *Let $k, r \in \mathbb{N}$, $k \le r-1$, $0 < \sigma \le \infty$, $\delta > 0$, and*

$$\widehat{\sigma} = \left(\frac{r}{k}\right)^{\frac{1}{2(r-k)}} \left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2r}}.$$

*Then*

$$E_\sigma(D^k, W_2^r(\mathbb{R}), \delta) = \begin{cases} \sigma^k \sqrt{\dfrac{r-k}{2\pi r} \left(\dfrac{k}{r}\right)^{\frac{k}{r-k}} \delta^2 + \dfrac{1}{\sigma^{2r}}}, & \sigma < \widehat{\sigma}, \\ \left(\dfrac{\delta^2}{2\pi}\right)^{\frac{r-k}{2r}}, & \sigma \ge \widehat{\sigma}, \end{cases}$$

*and the method*

(2)

$$\widehat{m}(y)(t) = \frac{1}{2\pi} \int_{|\tau| \le \sigma_0} (i\tau)^k \left(1 + \frac{r}{r-k} \left(\frac{r}{k}\right)^{\frac{k}{r-k}} \left(\frac{\tau}{\sigma_0}\right)^{2r}\right)^{-1} y(\tau) e^{i\tau t}\, d\tau,$$

*where $\sigma_0 = \min(\sigma, \widehat{\sigma})$, is optimal.*

*If $k = 0$ and $0 < \sigma < \infty$, then*

$$E_\sigma(D^k, W_2^r(\mathbb{R}), \delta) = \sqrt{\frac{\delta^2}{2\pi} + \frac{1}{\sigma^{2r}}}$$

*and the method*

$$\widehat{m}(y)(t) = \frac{1}{2\pi} \int_{|\tau| \le \sigma} \left(1 + \left(\frac{\tau}{\sigma}\right)^{2r}\right)^{-1} y(\tau) e^{i\tau t}\, d\tau$$

*is optmal.*

For a fixed error of input data consider the error of optimal recovery $E_\sigma$ as a function of $\sigma$. The larger interval $(-\sigma, \sigma)$ we take the less error we have. But beginning with $\widehat{\sigma}$ the error $E_\sigma$ does not change (see Fig. 1).

Consequently, for $\sigma > \widehat{\sigma}$ the observed information becomes partially redundant. To avoid this case the following condition

$$\delta^2 \sigma^{2r} \le 2\pi \left(\frac{r}{k}\right)^{\frac{r}{r-k}}$$

should hold. This inequality may be considered as some "uncertain principle".

There is another information characteristic besides $\widehat{\sigma}$. Further we will discuss it.

Usually numerical algorithms (for instance, interpolation or quadrature formulae) considered as good ones if they are exact for subspaces of algebraic or trigonometric polynomials. In this connection one tries to make the dimension of such space as large as possible. For example, Gauss quadrature is constructed to make the largest value of $n$ so that all polynomials of degree $n$ and below are integrated exactly. For

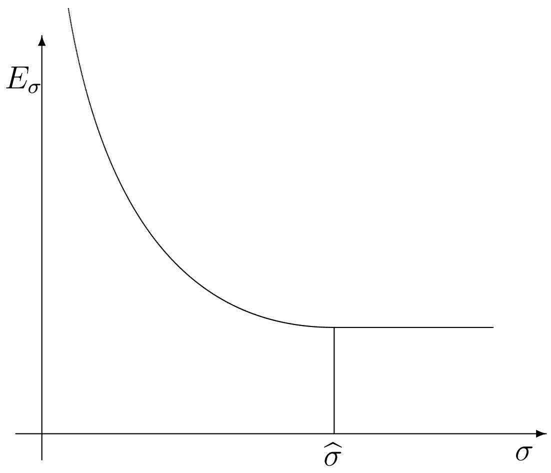


FIGURE 1

functions defined on $\mathbb{R}$ an analog of polynomials is the space of entire functions of exponential type.

Let $B_{\sigma',2}$ be the space of entire functions of exponential type $\sigma'$, such that their restriction on $\mathbb{R}$ belong to $L_2(\mathbb{R})$. It is appeared that there are many various optimal methods of recovery in problem (1) (for $k > 0$ method (2) is one of them). We say that a method $m$ is exact for a function $x(\cdot)$ if

$$m\left(Fx\big|_{\Delta_\sigma}\right)(\cdot) = x^{(k)}(\cdot).$$

We pose the problem to construct an optimal recovery method which will be exact for all functions from the space $B_{\sigma',2}$ for the largest value of $\sigma'$.

The same problem may be formulated in the equivalent form. Consider an extension of $W_2^r(\mathbb{R})$

$$W_{2,\sigma'}^r = W_2^r(\mathbb{R}) + B_{\sigma',2}.$$

Are there such of them that the error of optimal recovery for $W_{2,\sigma'}^r$ is the same as for $W_2^r(\mathbb{R})$? What is the largest extension of this type?

**Theorem 2.** *Let $k, r \in \mathbb{N}$, $k \le r-1$, $0 < \sigma \le \infty$, and $\delta > 0$. Set*

$$\widehat{\sigma}' = \left(\frac{r-k}{r}\right)^{\frac{1}{2k}} \left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2r}}, \quad \sigma_0' = \min\left\{\frac{\widehat{\sigma}'}{\widehat{\sigma}}\sigma, \widehat{\sigma}'\right\}.$$

*Then for all $0 \le \sigma' \le \sigma_0'$*

$$E_\sigma(D^k, W_{2,\sigma'}^r, \delta) = E_\sigma(D^k, W_2^r(\mathbb{R}), \delta).$$

*The method*

$$(3)\quad \widehat{m}(y)(t)=\frac{1}{2\pi}\int_{|\tau|\le\sigma_0'}(i\tau)^k y(\tau)e^{i\tau t}\,d\tau$$
$$+\frac{1}{2\pi}\int_{\sigma_0'\le|\tau|\le\sigma_0}(i\tau)^k\left(1+\frac{r}{r-k}\left(\frac{r}{k}\right)^{\frac{k}{r-k}}\left(\frac{\tau}{\sigma_0}\right)^{2r}\right)^{-1}y(\tau)e^{i\tau t}\,d\tau$$

*is optimal for the problem of optimal recovery of* $x^{(k)}(\cdot)$ *on the class* $W^r_{2,\sigma'}$ *for all* $0\le\sigma'\le\sigma_0'$.

Note that method (3) differs from (2) only by the fact that input data do not smooth on the interval $(-\sigma_0',\sigma_0')$. Nevertheless it has some good properties. Method (2) is exact for functions from $B_{\sigma_0',2}$ and optimal for the classes $W^r_{2,\sigma'}$ for all $0\le\sigma'\le\sigma_0'$.

Now we proceed with the general setting of optimal recovery problem. We formulate also a theorem which is the basic tool to construct optimal recovery methods.

Let $X$ be a linear space, $Y_1,\dots,Y_n$ be linear spaces with semi-inner products $(\cdot,\cdot)_{Y_j}$, $j=1,\dots,n$, and the corresponding semi-norms $\|\cdot\|_{Y_j}$ ($\|x\|_{Y_j}=\sqrt{(x,x)_{Y_j}}$), $I_j\colon X\to Y_j$, $j=1,\dots,n$, be linear operators, and $Z$ be a normed linear space. We consider the problem of optimal recovery of the operator $T\colon X\to Z$ on the set

$$W_k=\{\,x\in X\mid \|I_jx\|_{Y_j}\le\delta_j,\ 1\le j\le k,\ 0\le k<n\,\}$$

(for $k=0$ we take $W_0=X$) from the information about values of operators $I_{k+1},\dots,I_n$ given with errors. We assume that for any $x\in W$ we know the vector $y=(y_{k+1},\dots,y_n)$ such that

$$\|I_jx-y_j\|_{Y_j}\le\delta_j,\quad j=k+1,\dots,n.$$

Knowing the vector $y$ we want to recover $Tx$.

Here $\delta_j>0$, $j=1,\dots,k$, characterize a priory information about an element $x\in X$, and for $j=k+1,\dots,n$ they characterize a posteriori information about the same element. Further we will see that dual extremal problems connected with optimal recovery problems "do not distinguish" these two type of information. Thus it is convenient to have some symmetry in notation of these types of information.

Any operator $m\colon Y_{k+1}\times\ldots\times Y_n\to Z$ is admitted as a recovery method. The value

$$e(T,W_k,I,\delta,m)=\sup_{x\in W_k}\ \sup_{\substack{y=(y_{k+1},\dots,y_n)\in Y_{k+1}\times\ldots\times Y_n\\ \|I_jx-y_j\|_{Y_j}\le\delta_j,\ j=k+1,\dots,n}}\|Tx-m(y)\|_Z$$

is called the error of recovery of the method $m$ (here $I=(I_1,\dots,I_n)$, $\delta=(\delta_1,\dots\delta_n)$). We are interested in the value

$$E(T,W_k,I,\delta)=\inf_{m\colon Y_{k+1}\times\ldots\times Y_n\to Z}e(T,W_k,I,\delta,m)$$

which is called the error of optimal recovery. A method delivering the lower bound is called optimal.

The considered problem of optimal recovery is closely connected with the following extremal problem (we shall call it the duality extremal problem)

$$\|Tx\|_Z^2 \to \max, \quad \|I_jx\|_{Y_j}^2 \le \delta_j^2, \ j = 1,\dots,n, \ x \in X. \tag{4}$$

Now we formulate the main result.

**Theorem 3.** *Assume that there exist $\widehat{\lambda}_j \ge 0$, $j = 1,\dots,n$, such that the value of the extremal problem*

$$\|Tx\|_Z^2 \to \max, \quad \sum_{j=1}^n \widehat{\lambda}_j \|I_jx\|_{Y_j}^2 \le \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2, \quad x \in X, \tag{5}$$

*is the same as in* (4)*. Moreover, assume that for all $y = (y_1,\dots,y_n) \in Y_1 \times \ldots \times Y_n$ there exists $x_y = x(y_1,\dots,y_n)$ which is a solution of the extremal problem*

$$\sum_{j=1}^n \widehat{\lambda}_j \|I_jx - y_j\|_{Y_j}^2 \to \min, \quad x \in X.$$

*Then for all $k$, $0 \le k < n$,*

$$E(T, W_k, I, \delta) = \sup_{\substack{x\in X \\ \|I_jx\|_{Y_j} \le \delta_j, \ j=1,\dots,n}} \|Tx\|_Z$$

*and the method*

$$\widehat{m}(y_{k+1},\dots,y_n) = Tx(0,\dots,0,y_{k+1},\dots,y_n)$$

*is optimal.*

Now we apply these result to optimal recovery of solutions of evolutionary equations. Suppose that we can observe (with a known accuracy) the temperature of some object at the times $t_1,\dots,t_n$. What is the best possible way to use this information to recover the temperature of the object at the time $\tau \ne t_i$, $1 \le i \le n$?

The equation of heat-conduction for an infinite rod is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{6}$$

with the initial temperature distribution

$$u(0,x) = u_0(x). \tag{7}$$

We assume that $u_0(\cdot) \in L_2(\mathbb{R})$. The unique solution of problem (6)–(7) is the Poisson integral

$$u(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{4t}} u_0(\xi)\, d\xi, \quad t > 0. \tag{8}$$

Moreover, $u(t,\cdot) \to u_0(\cdot)$ in the $L_2(\mathbb{R})$-metric as $t \downarrow 0$.

We state the following problem. Suppose that we know temperature distributions $u(t_1,\cdot),\dots,u(t_n,\cdot)$ (at the times $0\le t_1<\dots<t_n$) with some accuracy. More precisely we know functions $y_i(\cdot)\in L_2(\mathbb{R})$, $i=1,\dots,n$, such that

$$\|u(t_i,\cdot)-y_i(\cdot)\|_{L_2(\mathbb{R})}\le\delta_i,$$

where $\delta_i>0$, $i=1,\dots,n$.

What is the best way to use this information to recover the temperature distribution of the rod at the time $\tau\ne t_i$, $1\le i\le n$, that is to recover the function $u(\tau,\cdot)$? It is more convenient to give the answer for the question "How to use the given information in the best way" in a quite more general situation.

Let $d$ be a natural number and $\psi(\cdot)$ be a continuous real function on $\mathbb{R}^d$ such that

$$\sup_{\xi\in\mathbb{R}^d}\psi(\xi)=+\infty\quad\text{and}\quad\inf_{\xi\in\mathbb{R}^d}\psi(\xi)=a>-\infty.$$

Set

$$L_2^\psi(\mathbb{R}^d)=\left\{x(\cdot)\in L_2(\mathbb{R}^d):\int_{\mathbb{R}^d}\psi^2(\xi)|Fx(\xi)|^2\,d\xi<\infty\right\},$$

where $F$ is the Fourier transform in $L_2(\mathbb{R}^d)$. Define the operator $A\colon L_2^\psi(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$ as follows

$$Ax(\cdot)=F^{-1}(\psi(\cdot)Fx(\cdot))(\cdot),$$

where $F^{-1}$ is the inverse Fourier transform.

Let $x_0(\cdot)\in L_2(\mathbb{R}^d)$. Consider the abstract Cauchy problem

$$\frac{dx}{dt}+Ax=0, \tag{9}$$

$$x\Big|_{t=0}=x_0(\cdot). \tag{10}$$

By the solution of this problem we mean the differential function $t\to x(t,\cdot)$ on $(0,\infty)$ with values in $L_2^\psi(\mathbb{R}^d)$, which satisfies equation (9) and $x(t,\cdot)\to x_0(\cdot)$ in $L_2(\mathbb{R}^d)$ as $t\downarrow 0$.

It is easy to check that the unique solution of problem (9)–(10) is the function

$$t\to P_t^\psi x_0(\cdot)=F^{-1}(e^{-\psi(\cdot)t}Fx_0(\cdot))(\cdot). \tag{11}$$

In particular, if $\psi(\xi)=|\xi|^\alpha$ where $|\xi|=\sqrt{\xi_1^2+\dots+\xi_d^2}$ and $\alpha>0$, then (11) is the solution of the generalize equation of heat-conduction

$$\frac{dx}{dt}+(-\Delta)^{\alpha/2}x=0,$$

$$x\Big|_{t=0}=x_0(\cdot).$$

If $\alpha=2$ and $d=1$, then (11) is the solution of (6)–(7).

We state the problem of optimal recovery of the solution of problem (9)–(10) as follows. Assume that we know approximate solutions of this problem at the times $0 \le t_1 < \ldots < t_n$, that is, we know functions $y_j(\cdot) \in L_2(\mathbb{R}^d)$, $j = 1, \ldots, n$, such that for some $x_0(\cdot) \in L_2(\mathbb{R}^d)$

$$\|P^{\psi}_{t_j} x_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_j, \quad j = 1, \ldots, n,$$

where $\delta_j > 0$, $j = 1, \ldots, n$. Using this information we have to recover the solution at the time $\tau \ne t_j$, that is, the function $P^{\psi}_{\tau} x_0(\cdot)$.

We consider arbitrary mappings $m\colon (L_2(\mathbb{R}^d))^n \to L_2(\mathbb{R}^d)$ as methods of recovery. The error of the method $m$ is defined as follows

$$e(\tau, A, \overline{\delta}, m) = \sup_{\substack{x_0(\cdot) \in L_2(\mathbb{R}^d), \overline{y}(\cdot) \in (L_2(\mathbb{R}^d))^n \\ \|P^{\psi}_{t_j} x_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_j,\ j=1,\ldots,n}} \|P^{\psi}_{\tau} x_0(\cdot) - m(\overline{y}(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)}$$

(here $\overline{\delta} = (\delta_1, \ldots, \delta_n)$, $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot))$).

We are interested in the value

$$E(\tau, A, \overline{\delta}) = \inf_{m\colon (L_2(\mathbb{R}^d))^n \to L_2(\mathbb{R}^d)} e(\tau, A, \overline{\delta}, m), \tag{12}$$

which is called the error of optimal recovery and in the method $\widehat{m}$, for which the infinum is attained that is in the method $\widehat{m}$ for which

$$E(\tau, A, \overline{\delta}) = e(\tau, A, \overline{\delta}, \widehat{m}).$$

We call this method the optimal recovery method.

Note that this approach was initiated by A. N. Kolmogorov who in 30's years of the previous century began to consider problems of the best tools of approximation for all functions from a given class of functions.

To formulate the result we give preliminary definitions. Consider the set

$$M = \operatorname{co}\{\, (t_j, \ln(1/\delta_j)),\ 1 \le j \le n \,\} + \{\, (t, at) \mid\ t \ge 0 \,\},$$

where $\operatorname{co} A$ is a convex hull of $A$. Define the function $\theta(\cdot)$ as follows

$$\theta(t) = \max\{\, x \mid (t, x) \in M \,\}.$$

It is clear that $\theta(\cdot)$ is a polygonal line on $[t_1, \infty)$ and its points of break $t_{s_1} < \ldots < t_{s_k}$ are a subset of $\{t_1, \ldots, t_n\}$ (see Fig. 2).

**Theorem 4.** *For all $\tau \ge 0$ the following equality*

$$E(\tau, A, \overline{\delta}) = e^{-\theta(\tau)}$$

*holds. If $t_{s_j} < \tau < t_{s_{j+1}}$, then the method*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = (K_{s_j} * y_{s_j})(\cdot) + (K_{s_{j+1}} * y_{s_{j+1}})(\cdot),$$

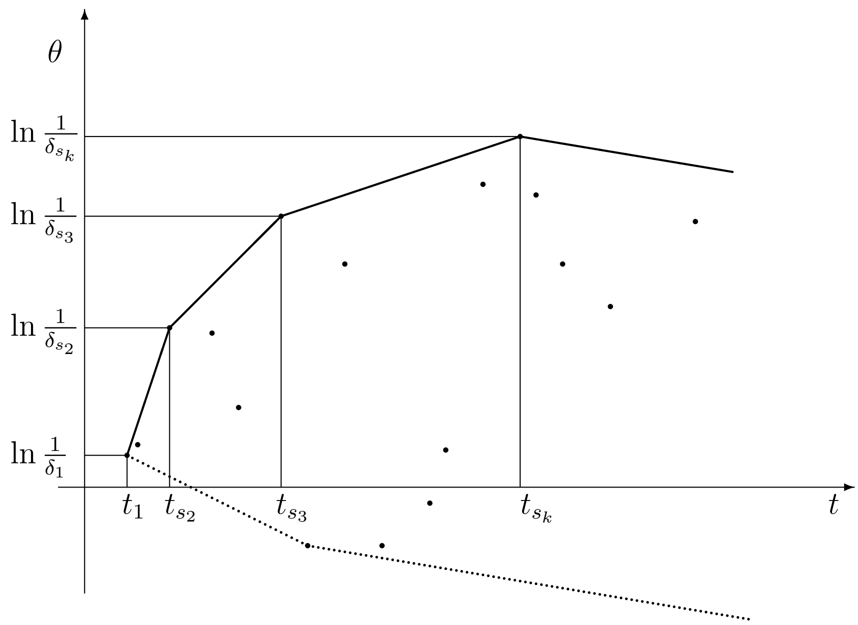


FIGURE 2

*is optimal; here*

$$FK_{s_j}(\xi) = \frac{\lambda_{s_j} e^{-\psi(\xi)(\tau - t_{s_j})}}{\lambda_{s_j} + \lambda_{s_{j+1}} e^{-2\psi(\xi)(t_{s_{j+1}} - t_{s_j})}} \ ,$$

$$FK_{s_{j+1}}(\xi) = \frac{\lambda_{s_{j+1}} e^{-\psi(\xi)(\tau + t_{s_{j+1}} - 2t_{s_j})}}{\lambda_{s_j} + \lambda_{s_{j+1}} e^{-2\psi(\xi)(t_{s_{j+1}} - t_{s_j})}},$$

$$\lambda_{s_j} = \frac{t_{s_{j+1}} - \tau}{t_{s_{j+1}} - t_{s_j}} \left( \frac{\delta_{s_{j+1}}}{\delta_{s_j}} \right)^{\frac{2(\tau - t_{s_j})}{t_{s_{j+1}} - t_{s_j}}} ,$$

$$\lambda_{s_{j+1}} = \frac{\tau - t_{s_j}}{t_{s_{j+1}} - t_{s_j}} \left( \frac{\delta_{s_j}}{\delta_{s_{j+1}}} \right)^{\frac{2(t_{s_{j+1}} - \tau)}{t_{s_{j+1}} - t_{s_j}}} .$$

*If $\tau > t_{s_k}$, then the method*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = P^{\psi}_{\tau - t_{s_k}} y_{s_k}(\cdot)$$

*is optimal.*

For $\psi(\xi) = |\xi|^2$ this Theorem is proved in [2].

We give the scheme of obtaining optimal recovery method of the solution for the abstract Cauchy problem. First we consider the dual extremal problem

$$\|P^{\psi}_{\tau}x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \max, \quad \|P^{\psi}_{t_j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \delta_j^2,\ j = 1, \dots, n,$$
$$x(\cdot) \in L_2(\mathbb{R}^d).$$

Passing to Fourier transforms and using the Plancherel theorem this problem can be rewritten in the form

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-2\psi(\xi)\tau}|Fx(\xi)|^2\,d\xi \to \max,$$
$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-2\psi(\xi)t_j}|Fx(\xi)|^2\,d\xi \le \delta_j^2, \quad j = 1, \dots, n, \quad x(\cdot) \in L_2(\mathbb{R}^d).$$

Since there is no existence of solution for this problem we consider the extension of it to the set of positive measures on $\mathbb{R}^d$, replacing $(2\pi)^{-d}|Fx_0(\xi)|^2\,d\xi$ by a positive measure $d\mu(\xi)$:

$$\int_{\mathbb{R}^d} e^{-2\psi(\xi)\tau}\,d\mu(\xi) \to \max,$$
$$\int_{\mathbb{R}^d} e^{-2\psi(\xi)t_j}\,d\mu(\xi) \le \delta_j^2, \quad j = 1, \dots, n, \quad d\mu(\xi) \ge 0. \tag{13}$$

For this problem the Lagrange function has the following form

$$\mathcal{L}(d\mu(\cdot), \lambda) = -\int_{\mathbb{R}^d} e^{-2\psi(\xi)\tau}\,d\mu(\xi) + \sum_{j=1}^n \lambda_j\left(\int_{\mathbb{R}^d} e^{-2\psi(\xi)t_j}\,d\mu(\xi) - \delta_j^2\right),$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$. We find the Lagrange multipliers $\widehat{\lambda}_1, \dots, \widehat{\lambda}_n$ (it appears that there are not more than two of them which are not vanishing). Then for the fixed functions $y_1(\cdot), \dots, y_n(\cdot)$ we consider the extremal problem

$$\sum_{j=1}^n \widehat{\lambda}_j\|P^{\psi}_{t_j}x(\cdot) - y_j(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \min \quad x(\cdot) \in L_2(\mathbb{R}^d). \tag{14}$$

The method

$$\widehat{m}(y) = P^{\psi}_t\widehat{x}(\cdot),$$

where $\widehat{x}(\cdot)$ is the solution of (14) is an optimal method of recovery.

Now let us consider the periodic case. Thus we consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{15}$$

with the initial data

$$u(t, 0) = u(t, \pi) = 0, \quad u(0, x) = u_0(x). \tag{16}$$

The solution of this problem is given by the series

$$u(t,x)=\sum_{k=1}^{\infty} b_k(u_0(\cdot))e^{-k^2t}\sin kx,$$

$$b_k(u_0(\cdot))=\frac{2}{\pi}\int_0^{\pi} u_0(x)\sin kx\,dx.$$

Denote by $W_2^r([0,\pi])$ the Sobolev class of functions on $[0,\pi]$:

$$W_2^r([0,\pi])=\{\,u(\cdot):u^{(r-1)}(\cdot)\text{ abs. cont. on }[0,\pi],\\ \|u^{(r)}(\cdot)\|_{L_2([0,\pi])}\le 1\,\}.$$

We are interested in the recovery of the solution of (15)–(16) at some fixed time $\tau$, provided that $u_0(\cdot)\in W_2^r([0,\pi])$ and we know the vector $\overline{b}(u_0(\cdot))=(b_1(u_0(\cdot)),\dots,b_n(u_0(\cdot)))$ of the first $n$ Fourier coefficients of $u_0(\cdot)$ with some accuracy $\delta$, namely, we know a vector $\overline{y}=(y_1,\dots,y_n)$ for which

$$\|\overline{b}(u_0(\cdot))-\overline{y}\|_{l_2^n}=\sqrt{\sum_{k=1}^{n}|b_k(u_0(\cdot))-y_k|^2}\le\delta.$$

As above we consider arbitrary mappings $m\colon\mathbb{R}^n\to L_2([0,\pi])$ as methods of recovery. The error of the method $m$ is defined as follows

$$e(\tau,W_2^r([0,\pi]),n,\delta,m)=\sup_{\substack{u_0(\cdot)\in W_2^r([0,\pi]),\ \overline{y}\in\mathbb{R}^n\\ \|\overline{b}(u_0(\cdot))-\overline{y}\|_{l_2^n}\le\delta}}\|u(\tau,\cdot)-m(\overline{y})(\cdot)\|_{L_2([0,\pi])}.$$

We are interested in the value

$$E(\tau,W_2^r([0,\pi]),n,\delta)=\inf_{m\colon\mathbb{R}^n\to L_2([0,\pi])} e(\tau,W_2^r([0,\pi]),n,\delta,m),\tag{17}$$

which is called the error of optimal recovery and in the optimal method $\widehat{m}$, for which the infinum is attained.

The following result is obtained in [3].

**Theorem 5.** *If $0<\delta<1$, then*

$$E(\tau,W_2^r([0,\pi]),n,\delta)=e^{-\tau}\sqrt{\delta^2+\frac{1-\delta^2}{(n+1)^{2r}}e^{-2\tau n(n+2)}}$$

*and the method*

$$\widehat{m}(\overline{y})(x)=\sum_{k=1}^{n}\left(1+\frac{k^{2r}}{(n+1)^{2r}e^{2\tau n(n+2)}-1}\right)^{-1}y_ke^{-k^2\tau}\sin kx$$

*is optimal. If $\delta\ge 1$, then*

$$E(\tau,W_2^r([0,\pi]),n,\delta)=e^{-\tau}$$

*and $\widehat{m}(\overline{y})(\cdot)=0$ is an optimal method.*

To find an optimal method of recovery for problem (17) we consider the dual problem

$$\|u(\tau,\cdot)\|^2_{L_2([0,\pi])} \to \max, \quad \|\overline{b}(u_0(\cdot))\|^2_{l_2^n} \le \delta^2, \quad \|u_0^{(r)}(\cdot)\|^2_{L_2([0,\pi])} \le 1$$

Using Parseval's identity, this problem can be rewritten as

$$\sum_{k=1}^{\infty} b_k^2(u_0(\cdot))e^{-2k^2\tau} \to \max, \quad \sum_{k=1}^{n} b_k^2(u_0(\cdot)) \le \delta^2, \quad \sum_{k=1}^{\infty} b_k^2(u_0(\cdot))k^{2r} \le 1.$$

Denoting by $u_k = b_k(u_0(\cdot))$, we obtain the following problem of linear programming

$$\sum_{k=1}^{\infty} u_k e^{-2k^2\tau} \to \max, \quad \sum_{k=1}^{n} u_k \le \delta^2, \quad \sum_{k=1}^{\infty} u_k k^{2r} \le 1, \quad u_k \ge 0.$$

It is easy to find the solution of this problem and the corresponding Lagrange multipliers $\widehat{\lambda}_1$, $\widehat{\lambda}_2$. Then for a fixed vector $\overline{y} = (y_1, \dots, y_n)$ we consider the extremal problem

$$\widehat{\lambda}_1\|\overline{b}(u(\cdot)) - \overline{y}\|^2_{l_2^n} + \widehat{\lambda}_2\|u^{(r)}(\cdot)\|^2_{L_2([0,\pi])} \to \min.$$

Let $\widehat{u}(\cdot)$ be the solution of this problem. Then the method

$$m(\overline{y})(\cdot) = \sum_{k=1}^{\infty} b_k(\widehat{u}(\cdot))e^{-k^2\tau}\sin kx$$

is optimal.

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