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Optimal recovery and generalized Carlson inequality for weights with symmetry properties ☆

K.Yu. Osipenko

Moscow State University, Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russian Federation

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ABSTRACT

The paper concerns problems of the recovery of operators from noisy information in weighted L_q-spaces with homogeneous weights. A number of general theorems are proved and applied to finding exact constants in multidimensional Carlson type inequalities with several weights and problems of the recovery of differential operators from a noisy Fourier transform. In particular, optimal methods are obtained for the recovery of powers of generalized Laplace operators from a noisy Fourier transform in the L_p-metric.

1. Introduction

Let T be a non-empty set, Σ be the σ-algebra of subsets of T, and μ be a nonnegative σ-additive measure on Σ. We denote by $L_p(T, \Sigma, \mu)$ (or simply $L_p(T, \mu)$) the set of all Σ-measurable functions with values in $\mathbb{R}$ or in $\mathbb{C}$ for which

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E-mail address: kosipenko@yahoo.com.

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$$\|x(\cdot)\|_{L_p(T,\mu)} = \left(\int_T |x(t)|^p \, d\mu(t) \right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|x(\cdot)\|_{L_\infty(T,\mu)} = \operatorname*{vraisup}_{t\in T} |x(t)| < \infty, \quad p = \infty.$$

If $T \subset \mathbb{R}^d$ and $d\mu = dt$, $t \in \mathbb{R}^d$, we put $L_p(T) = L_p(T,\mu)$.

In this paper we study the extremal problem

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \to \max, \quad \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu))} \le \delta, \\ \|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1, \ j = 1, \dots, n, \quad (1)$$

where $w(\cdot)$, $w_0(\cdot)$, and $w_j(\cdot)$, $j = 1, \dots, n$, are homogenous functions with some symmetry properties. Using the solution of this problem we obtain the sharp constant K for the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le K \|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T,\mu))} \max_{1\le j\le n} \|w_j(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)}. \quad (2)$$

In particular, we find the sharp constant for the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d_+)} \le C \|w_0(\cdot)x(\cdot)\|^{p\alpha}_{L_p(\mathbb{R}^d_+)} \max_{1\le j\le d} \|w_j(\cdot)x(\cdot)\|^{r\beta}_{L_r(\mathbb{R}^d_+)},$$

where $w(t) = (t_1^2 + \dots + t_d^2)^{\theta/2}$, $w_0(t) = (t_1^2 + \dots + t_d^2)^{\theta_0/2}$, $w_j(t) = t_j^{\theta_1}$, $j = 1, \dots, d$, $\theta = d(1 - 1/q)$, $\theta_0 = d - (\lambda + d)/p$, $\theta_1 = d + (\mu - d)/r$,

$$\alpha = \frac{\mu}{p\mu + r\lambda}, \quad \beta = \frac{\lambda}{p\mu + r\lambda}, \quad \lambda, \mu > 0,$$

and $(p,q,r) \in P \cup P_1 \cup P_2$, where

$$P = \{\, (p,q,r) : 1 \le q < p, r \,\}, \quad P_1 = \{\, (p,q,r) : 1 \le q = r < p \,\}, \\ P_2 = \{\, (p,q,r) : 1 \le q = p < r \,\}.$$

For $d = 1$, $q = 1$, and $(p, 1, r) \in P$ this result was proved in [5] (see also [2]).

The value of extremal problem (1) is the error of optimal recovery of the operator $\Lambda x(\cdot) = w(\cdot)x(\cdot)$ on the class of functions $x(\cdot)$ such that $\|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1$, $j = 1, \dots, n$, by the information about the function $w_0(\cdot)x(\cdot)$ given with the error δ in L_p-norm. Therefore, we begin with the setting of optimal recovery problem and prove some general theorems concerning this problem.

Inequality (2) may be considered as generalization of the Carlson inequality [3]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le \sqrt{\pi} \|x(t)\|^{1/2}_{L_2(\mathbb{R}_+)} \|tx(t)\|^{1/2}_{L_2(\mathbb{R}_+)}, \quad \mathbb{R}_+ = [0, +\infty),$$

which was generalized by many authors (see [5], [1], [2], [4], [6], [9], [10]). In [9] we found sharp constants for inequalities of the form

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le K \|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T,\mu))} \|w_1(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)},$$

where T is a cone in a linear space, $w(\cdot)$, $w_0(\cdot)$, and $w_1(\cdot)$ are homogenous functions and $1 \le q < p, r < \infty$ (for $T = \mathbb{R}^d$ the sharp inequality was obtained in [2]).

The main difference of the results of this article and the previous ones (in particular, articles [2] and [9]) that we obtain here exact inequalities of Carlson type and optimal recovery methods for several weights. Moreover, we extend these results from the set P to the set $P \cup P_1 \cup P_2$.

We divide the article on two sections. The first section is devoted to the main general results and the second section is devoted to the application of obtained results to differential operators defined by Fourier transforms and examples of exact inequalities for such operators.

2. General setting and main results

Let T_0 be a non-empty μ-measurable subset of T. Put

$$\mathcal{W}=\{x(\cdot):x(\cdot)\in L_p(T_0,\mu),\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}<\infty,\ j=1,\dots,n\},$$
$$W=\{x(\cdot)\in\mathcal{W}:\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n\},$$

where $1\le p,r\le\infty$, and $\varphi_j(\cdot)$ is a measurable function on T. Consider the problem of recovery of operator $\Lambda\colon\mathcal{W}\to L_q(T,\mu)$, $1\le q\le\infty$, defined by equality $\Lambda x(\cdot)=\psi(\cdot)x(\cdot)$, where $\psi(\cdot)$ is a measurable function on T, on the class W by the information about functions $x(\cdot)\in W$ given inaccurately (we assume that $\psi(\cdot)$ and $\varphi_j(\cdot)$, $j=1,\dots,n$, such that Λ maps $\mathcal{W}$ to $L_q(T,\mu)$). More precisely, we assume that for any function $x(\cdot)\in W$ we know $y(\cdot)\in L_p(T_0,\mu)$ such that $\|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta$, $\delta>0$. We want to approximate the value $\Lambda x(\cdot)$ knowing $y(\cdot)$. As recovery methods we consider all possible mappings $m\colon L_p(T_0,\mu)\to L_q(T,\mu)$. The error of a method m is defined as

$$e(p,q,r,m)=\sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu)\\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}}\|\Lambda x(\cdot)-m(y)(\cdot)\|_{L_q(T,\mu)}.$$

The quantity

$$E(p,q,r)=\inf_{m\colon L_p(T_0,\mu)\to L_q(T,\mu)}e(p,q,r,m) \tag{3}$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal. Various settings of optimal recovery theory and examples of such problems may be found in [7], [13], [12].

For the lower bound of $E(p,q,r)$ we use the following result which was proved (in more or less general forms) in many papers (see, for example, [8]).

Lemma 1.

$$E(p,q,r)\ge\sup_{\substack{x(\cdot)\in W\\ \|x(\cdot)\|_{L_p(T_0,\mu)}\le\delta}}\|\Lambda x(\cdot)\|_{L_q(T,\mu)}. \tag{4}$$

Set

$$\chi_0(t)=\begin{cases}1, & t\in T_0,\\ 0, & t\notin T_0,\end{cases}\qquad \sigma_r(t)=\sum_{j=1}^n\lambda_j|\varphi_j(t)|^r.$$

Theorem 1. *Let $1\le q<p,r$, $\lambda_j\ge 0$, $j=0,1,\dots,n$, $\lambda_0+\sigma_r(t)\ne 0$ for almost all $t\in T_0$, $\sigma_r(t)\ne 0$ for almost all $t\in T\setminus T_0$, $\widehat{x}(t)\ge 0$ be a solution of equation*

$$-q|\psi(t)|^q+p\lambda_0x^{p-q}(t)\chi_0(t)+r\sigma_r(t)x^{r-q}(t)=0, \tag{5}$$

$\overline{\lambda}$ such that

$$\int_{T_0}\widehat{x}^p(t)\,d\mu(t)\le\delta^p,\quad \int_T|\varphi_j(t)|^r\widehat{x}^r(t)\,d\mu(t)\le 1,\ j=1,\dots,n,$$
$$\lambda_0\left(\int_{T_0}\widehat{x}^p(t)\,d\mu(t)-\delta^p\right)=0,\quad \lambda_j\left(\int_T|\varphi_j(t)|^r\widehat{x}^r(t)\,d\mu(t)-1\right)=0,$$
$$j=1,\dots,n. \tag{6}$$

Then

$$E(p,q,r)=\left(q^{-1}p\lambda_0\delta^p+q^{-1}r\sum_{j=1}^{n}\lambda_j\right)^{1/q}, \tag{7}$$

and the method

$$\widehat{m}(y)(t)=\begin{cases}q^{-1}p\lambda_0\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t), & t\in T_0,\ \psi(t)\neq 0,\\ 0, & \textit{otherwise},\end{cases} \tag{8}$$

is optimal recovery method.

To prove this theorem we need some preliminary results. The first one is actually a sufficient condition in the Kuhn-Tucker theorem (the only difference is that we do not require convexity of functions).

Let $f_j\colon A\to\mathbb{R}$, $j=0,1,\dots,k$, be functions defined on some set A. Consider the extremal problem

$$f_0(x)\to\max,\quad f_j(x)\le 0,\quad j=1,\dots,k,\quad x\in A, \tag{9}$$

and write down its Lagrange function

$$\mathcal{L}(x,\lambda)=-f_0(x)+\sum_{j=1}^{k}\lambda_j f_j(x),\quad \lambda=(\lambda_1,\dots,\lambda_k).$$

Lemma 2. *Assume that there exist $\widehat{\lambda}_j\ge 0$, $j=1,\dots,k$, and an element $\widehat{x}\in A$, admissible for problem (9), such that*

(a) $\min_{x\in A}\mathcal{L}(x,\widehat{\lambda})=\mathcal{L}(\widehat{x},\widehat{\lambda})$, $\widehat{\lambda}=(\widehat{\lambda}_1,\dots,\widehat{\lambda}_k)$,

(b) $\widehat{\lambda}_j f_j(\widehat{x})=0$, $j=1,\dots,k$ *(complementary slackness conditions).*

Then $\widehat{x}$ is an extremal element for problem (9).

Proof. For any x admissible for problem (9) we have

$$-f_0(x)\ge\mathcal{L}(x,\widehat{\lambda})\ge\mathcal{L}(\widehat{x},\widehat{\lambda})=-f_0(\widehat{x}).\quad\square$$

Note that conditions (6) are actually complementary slackness conditions.

Put

$$F(u,v,\alpha)=-((1-\alpha)u+\alpha v)^q+av^p+bu^r,\quad u,v\ge 0,\quad \alpha\in[0,1],$$

where $a,b\ge 0$, and $1\le p,q,r<\infty$.

Lemma 3 *([9]). For all $a,b\ge 0$, $a+b>0$, and all $1\le q<p,r<\infty$, there exists the unique solution $\widehat{u}>0$ of the equation*

$$-q+pau^{p-q}+rbu^{r-q}=0.$$

Moreover, for all $u,v\ge 0$ and $\alpha=q^{-1}pa\widehat{u}^{p-q}=1-q^{-1}rb\widehat{u}^{r-q}$

$$F(\widehat{u},\widehat{u},\alpha)\le F(u,v,\alpha).$$

In particular, for all $u\ge 0$

$$-\widehat{u}^q+a\widehat{u}^p+b\widehat{u}^r\le -u^q+au^p+bu^r.$$

Proof of Theorem 1. 1. Lower estimate. The extremal problem on the right-hand side of (4) (for convenience, we raise the quantity to be maximized to the q-th power) is as follows:

$$\int_T |\psi(t)x(t)|^q\,d\mu(t)\to\max,\quad \int_{T_0}|x(t)|^p\,d\mu(t)\le\delta^p,$$
$$\int_T|\varphi_j(t)x(t)|^r\,d\mu(t)\le 1,\ j=1,\dots,n. \tag{10}$$

If $t\in T$ such that $\psi(t)=0$, then evidently $\widehat{x}(t)=0$. If $\psi(t)\ne 0$ we obtain by Lemma 3 that there is the unique solution $\widehat{x}(t)$ of (5). It follows by (6) that $\widehat{x}(\cdot)$ is admissible function for problem (10). Therefore, by (4) we obtain

$$E(p,q,r)\ge\left(\int_T|\psi(t)|^q\widehat{x}^q(t)\,d\mu(t)\right)^{1/q}.$$

From (5) we have

$$|\psi(t)|^q\widehat{x}^q(t)=q^{-1}p\lambda_0\widehat{x}^p(t)\chi_0(t)+q^{-1}r\sigma_r(t)\widehat{x}^r(t).$$

Integrating this equality over the set T, we obtain

$$\int_T|\psi(t)|^q\widehat{x}^q(t)\,d\mu(t)=q^{-1}p\lambda_0\delta^p+q^{-1}r\sum_{j=1}^n\lambda_j.$$

Thus,

$$E(p,q,r)\ge\left(q^{-1}p\lambda_0\delta^p+q^{-1}r\sum_{j=1}^n\lambda_j\right)^{1/q}.$$

2. Upper estimate. To estimate the error of method (8) we need to find the value of the extremal problem:

$$\int_{T_0}|\psi(t)x(t)-\psi(t)\alpha(t)y(t)|^q\,d\mu(t)+\int_{T\setminus T_0}|\psi(t)x(t)|^q\,d\mu(t)\to\max,$$
$$\int_{T_0}|x(t)-y(t)|^p\,d\mu(t)\le\delta^p,\quad\int_T|\varphi_j(t)x(t)|^r\,d\mu(t)\le 1,\ j=1,\dots,n, \tag{11}$$

where

$$\alpha(t)=\begin{cases}q^{-1}p\lambda_0\widehat{x}^{p-q}(t)|\psi(t)|^{-q}, & t\in T_0,\ \psi(t)\ne 0,\\ 0, & \text{otherwise.}\end{cases}$$

Put

$$z(t)=\begin{cases}x(t)-y(t), & t\in T_0,\\ 0, & t\in T\setminus T_0.\end{cases}$$

Then (11) may be rewritten as follows:

$$\int_T |\psi(t)|^q |(1-\alpha(t))x(t)+\alpha(t)z(t)|^q\,d\mu(t) \to \max,$$

$$\int_{T_0} |z(t)|^p\,d\mu(t) \le \delta^p, \quad \int_T |\varphi_j(t)x(t)|^r\,d\mu(t) \le 1, \ j=1,\dots,n. \quad (12)$$

Since $|(1-\alpha(t))x(t)+\alpha(t)z(t)| \le (1-\alpha(t))|x(t)|+\alpha(t)|z(t)|$ the value of this problem does not exceed the value of the problem (we put $u(t)=|x(t)|$ and $v(t)=|z(t)|$)

$$\int_T |\psi(t)|^q ((1-\alpha(t))u(t)+\alpha(t)v(t))^q\,d\mu(t) \to \max,$$

$$\int_{T_0} v^p(t)\,d\mu(t) \le \delta^p, \quad \int_T |\varphi_j(t)|^r u^r(t)\,d\mu(t) \le 1, \ j=1,\dots,n,$$

$$u(t)\ge 0, \ v(t)\ge 0 \quad \text{for almost all } t\in T. \quad (13)$$

For every $u(\cdot)$ and $v(\cdot)$ admissible in (13) we can consider $x(\cdot)=u(\cdot)$ and $z(\cdot)=v(\cdot)$ which will be admissible in (12). Thus, the values of (12) and (13) are the same.

The Lagrange function for problem (13) has the following form:

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda}) = \int_T L(t,u(t),v(t),\overline{\lambda})\,d\mu(t),$$

where

$$L(t,u,v,\overline{\lambda}) = -|\psi(t)|^q((1-\alpha(t))u+\alpha(t)v)^q + \lambda_0 v^p \chi_0(t) + \sigma_r(t)u^r.$$

By Lemma 3 we have

$$L(t,\widehat{x}(t),\widehat{x}(t),\overline{\lambda}) \le L(t,u(t),v(t),\overline{\lambda}).$$

Thus,

$$\mathcal{L}(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda}) \le \mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda}).$$

It follows by Lemma 2 that functions $u(\cdot)=v(\cdot)=\widehat{x}(\cdot)$ are extremal in (13). Consequently,

$$e(p,q,r,\widehat{m}) \le \left(\int_T |\psi(t)|^q \widehat{x}^q(t)\,d\mu(t)\right)^{1/q} = \left(q^{-1}p\lambda_0\delta^p + q^{-1}r\sum_{j=1}^n \lambda_j\right)^{1/q} \le E(p,q,r).$$

It means that method (8) is optimal and equality (7) holds. □

Denote $a_+=\max\{a,0\}$.

Theorem 2. *Let* $1\le q=r<p$, $\lambda_0>0$, $\lambda_j\ge 0$, $j=1,\dots,n$,

$$\widehat{x}(t) = \begin{cases} \left(\dfrac{q}{p\lambda_0}\left(|\psi(t)|^q-\sigma_q(t)\right)_+\right)^{\frac{1}{p-q}}, & t\in T_0, \\ 0, & t\notin T_0, \end{cases} \quad (14)$$

$\overline{\lambda}$ *satisfies conditions* (6), *and* $|\psi(t)|^q-\sigma_q(t)\le 0$ *for almost all* $t\notin T_0$. *Then*

$$E(p,q,q) = \left(q^{-1}p\lambda_0\delta^p + \sum_{j=1}^n \lambda_j\right)^{1/q}, \quad (15)$$

and the method

$$\widehat{m}(y)(t)=\begin{cases}\left(1-|\psi(t)|^{-q}\sigma_q(t)\right)_+\psi(t)y(t), & t\in T_0,\ \psi(t)\neq 0,\\ 0, & \textit{otherwise},\end{cases}\tag{16}$$

is optimal.

Proof. 1. Lower estimate. It follows by (6) that $\widehat{x}(\cdot)$ is admissible function for extremal problem in the right-hand side of (4). Therefore,

$$E(p,q,q)\ge\left(\int_T|\psi(t)|^q\widehat{x}^q(t)\,d\mu(t)\right)^{1/q}.$$

From the definition of $\widehat{x}(\cdot)$ we have

$$|\psi(t)|^q\widehat{x}^q(t)=q^{-1}p\lambda_0\widehat{x}^p(t)\chi_0(t)+\sigma_q(t)\widehat{x}^q(t).$$

Integrating this equality, we obtain

$$\int_T|\psi(t)|^q\widehat{x}^q(t)\,d\mu(t)=q^{-1}p\lambda_0\delta^p+\sum_{j=1}^n\lambda_j.$$

Thus,

$$E(p,q,q)\ge\left(q^{-1}p\lambda_0\delta^p+\sum_{j=1}^n\lambda_j\right)^{1/q}.$$

2. Upper estimate. Put

$$\alpha(t)=\begin{cases}\left(1-|\psi(t)|^{-q}\sigma_q(t)\right)_+, & t\in T_0,\ \psi(t)\neq 0,\\ 0, & \text{otherwise}.\end{cases}$$

To estimate the error of method (16) we need to find the value of the extremal problem:

$$\int_{T_0}|\psi(t)|^q|x(t)-\alpha(t)y(t)|^q\,d\mu(t)+\int_{T\setminus T_0}|\psi(t)x(t)|^q\,d\mu(t)\to\max,$$
$$\int_{T_0}|x(t)-y(t)|^p\,d\mu(t)\le\delta^p,\quad\int_T|\varphi_j(t)x(t)|^q\,d\mu(t)\le 1,\ j=1,\dots,n.$$

Putting $z(\cdot)=x(\cdot)-y(\cdot)$ this problem may be rewritten in the following form

$$\int_{T_0}|\psi(t)|^q|(1-\alpha(t))x(t)+\alpha(t)z(t)|^q\,d\mu(t)+\int_{T\setminus T_0}|\psi(t)x(t)|^q\,d\mu(t)\to\max,$$
$$\int_{T_0}|z(t)|^p\,d\mu(t)\le\delta^p,\quad\int_T|\varphi_j(t)x(t)|^q\,d\mu(t)\le 1,\ j=1,\dots,n.$$

By the same arguments that were given above the value of this problem coincides with the value of the problem

$$\int_T |\psi(t)|^q((1-\alpha(t))v(t)+\alpha(t)u(t))^q\,d\mu(t)\to\max,$$

$$\int_{T_0} u^p(t)\,d\mu(t)\le\delta^p,\quad \int_T |\varphi_j(t)|^q v^q(t)\,d\mu(t)\le 1,\ j=1,\dots,n,$$

$$u(t),\,v(t)\ge 0,\ \text{for almost all } t\in T. \tag{17}$$

The Lagrange function of (17) has the form

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})=\int_T L(u(t),v(t),\overline{\lambda})\,d\mu(t),$$

where

$$L(u,v,\overline{\lambda})=\begin{cases}-|\psi(t)|^q((1-\alpha(t))v+\alpha(t)u)^q+\lambda_0u^p+\sigma_q(t)v^q, & t\in T_0,\\ -|\psi(t)|^q v^q+\sigma_q(t)v^q, & t\notin T_0.\end{cases}$$

If $\alpha(t)>0$, then

$$\frac{\partial L}{\partial v}=q(v^{q-1}-((1-\alpha(t))v+\alpha(t)u)^{q-1})\sigma_q(t).$$

Therefore, for $\alpha(t)>0$ and any $u>0$, the function $L(u,v,\overline{\lambda})$, $v\in(0,+\infty)$, reaches a minimum at $v=u$. Set $T_0'=\{t\in T_0:\alpha(t)>0\}$. We have

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})\ge\int_{T_0'} L(u(\cdot),u(\cdot),\overline{\lambda})\,d\mu(t).$$

It is easily checked that for $t\in T_0'$ for all $u(t)\ge 0$

$$L(u(\cdot),u(\cdot),\overline{\lambda})\ge L(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda}).$$

Consequently,

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})\ge\int_{T_0'} L(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda})\,d\mu(t)=\mathcal{L}(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda}).$$

Taking into account (6) we obtain by Lemma 2 that $u(\cdot)=v(\cdot)=\widehat{x}(\cdot)$ are extremal functions in (17). Thus,

$$e^q(p,q,q,\widehat{m})=\int_T|\psi(t)\widehat{x}(t)|^q\,d\mu(t)=q^{-1}p\lambda_0\delta^p+\sum_{j=1}^n\lambda_j\le E^q(p,q,q).$$

It means that the method $\widehat{m}$ is optimal and the optimal recovery error is as stated. □

Theorem 3. *Let $1\le q=p<r$, $\lambda_0>0$, $\lambda_j\ge 0$, $j=1,\dots,n$, $\sigma_r(t)\ne 0$ for almost all $t\in T$,*

$$\widehat{x}(t)=\begin{cases}\left(pr^{-1}\sigma_r^{-1}(t)(|\psi(t)|^p-\lambda_0)_+\right)^{\frac{1}{r-p}}, & t\in T_0,\\ \left(pr^{-1}\sigma_r^{-1}(t)|\psi(t)|^p\right)^{\frac{1}{r-p}}, & t\in T\setminus T_0,\end{cases} \tag{18}$$

and $\overline{\lambda}$ satisfies conditions (6). Then

$$E(p,p,r)=\left(\lambda_0\delta^p+\frac{r}{p}\sum_{j=1}^n\lambda_j\right)^{1/p}, \tag{19}$$

and the method

$$\widehat{m}(y)(t) = \begin{cases} \alpha(t)\psi(t)y(t), & t \in T_0, \\ 0, & t \in T \setminus T_0, \end{cases} \tag{20}$$

where

$$\alpha(t) = \begin{cases} \min\left\{1, \lambda_0|\psi(t)|^{-p}\right\}, & t \in T_0,\ \psi(t) \neq 0, \\ 0, & \textit{otherwise}, \end{cases}$$

is optimal.

Proof. 1. Lower estimate. By the definition of $\widehat{x}(\cdot)$ we have

$$|\psi(t)|^p \widehat{x}^p(t) = \lambda_0 \widehat{x}^p(t)\chi_0(t) + \frac{r}{p}\sigma_r(t)\widehat{x}^r(t).$$

Using the similar arguments as in the proof of Theorem 1 we obtain

$$E(p,p,r) \geq \left(\int_T |\psi(t)|^p \widehat{x}^p(t)\,d\mu(t)\right)^{1/p} = \left(\lambda_0\delta^p + \frac{r}{p}\sum_{j=1}^n \lambda_j\right)^{1/p}.$$

2. Upper estimate. To estimate the error of method (20) we need to find the value of the following extremal problem:

$$\int_{T_0} |\psi(t)|^p |x(t) - \alpha(t)y(t)|^p\,d\mu(t) + \int_{T\setminus T_0} |\psi(t)x(t)|^p\,d\mu(t) \to \max,$$

$$\int_{T_0} |x(t) - y(t)|^p\,d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^r\,d\mu(t) \leq 1,\ j = 1,\dots,n.$$

Putting $z(\cdot) = x(\cdot) - y(\cdot)$ this problem may be rewritten in the form

$$\int_{T_0} |\psi(t)|^p |(1-\alpha(t))x(t) + \alpha(t)z(t)|^p\,d\mu(t) + \int_{T\setminus T_0} |\psi(t)x(t)|^p\,d\mu(t) \to \max,$$

$$\int_{T_0} |z(t)|^p\,d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^r\,d\mu(t) \leq 1,\ j = 1,\dots,n.$$

The value of this problem coincides with the value of the problem

$$\int_T |\psi(t)|^p ((1-\alpha(t))v(t) + \alpha(t)u(t))^p\,d\mu(t) \to \max,$$

$$\int_{T_0} u^p(t)\,d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)|^r v^r(t)\,d\mu(t) \leq 1,\ j = 1,\dots,n,$$

$$u(t), v(t) \geq 0,\ \text{for almost all } t \in T. \tag{21}$$

The Lagrange function of (21) has the form

$$\mathcal{L}(u(\cdot), v(\cdot), \overline{\lambda}) = \int_T L(u(t), v(t), \overline{\lambda})\,d\mu(t),$$

where

$$L(u,v,\overline{\lambda})=\begin{cases}-|\psi(t)|^p((1-\alpha(t))v+\alpha(t)u)^p+\lambda_0u^p+\sigma_r(t)v^r, & t\in T_0,\\ -|\psi(t)|^pv^p+\sigma_r(t)v^r, & t\in T\setminus T_0.\end{cases}$$

For $t\in T_0$ and $|\psi(t)|^p>\lambda_0$ we have

$$\frac{\partial L}{\partial u}=p\lambda_0(u^{p-1}-((1-\alpha(t))v+\alpha(t)u)^{p-1}).$$

Consequently, in this case for any $v>0$ the function $L(u,v,\overline{\lambda})$, $v\in(0,+\infty)$, reaches a minimum at $v=u$. If $t\in T_0$, $0<|\psi(t)|^p\le\lambda_0$, then $\alpha(t)=1$ and $L(u,v,\overline{\lambda})\ge 0$. If $t\in T_0$ and $\psi(t)=0$, then again $L(u,v,\overline{\lambda})\ge 0$. Set $T_1=\{t\in T_0:|\psi(t)|^p>\lambda_0\}$. Then for all $u(t),v(t)\ge 0$ we have

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})\ge\int_{T_1}L(v(\cdot),v(\cdot),\overline{\lambda})\,d\mu(t)+\int_{T\setminus T_0}L(v(\cdot),v(\cdot),\overline{\lambda})\,d\mu(t).$$

It is easy to check that for all $v(t)\ge 0$

$$L(v(\cdot),v(\cdot),\overline{\lambda})\ge L(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda}).$$

Therefore,

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})\ge\int_{T_1\cup(T\setminus T_0)}L(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda})\,d\mu(t)=\mathcal{L}(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda}).$$

Taking into account (6) we obtain by Lemma 2 that $u(\cdot)=v(\cdot)=\widehat{x}(\cdot)$ are extremal functions in (21). Consequently,

$$e^p(p,p,r,\widehat{m})=\int_T|\psi(t)\widehat{x}(t)|^q\,d\mu(t)=\lambda_0\delta^p+\frac{r}{p}\sum_{j=1}^n\lambda_j\le E^p(p,p,r).$$

It means that the method $\widehat{m}$ is optimal and the optimal recovery error is as stated. □

Note that if conditions of Theorems 1, 2, and 3 are fulfilled, then we have

$$E(p,q,r)=\sup_{\substack{\|x(\cdot)\|_{L_p(T_0,\mu)}\le\delta\\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n}}\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}. \tag{22}$$

In Theorems 1, 2, and 3 we actually used the Lagrange principle and expressed the solutions in terms of the extremal function $\widehat{x}(\cdot)$ and Lagrange multipliers λ_j, $j=0,1,\dots,n$. Next, we will consider the cases when the extremal function $\widehat{x}(\cdot)$ and Lagrange multipliers λ_j, $j=0,1,\dots,n$, can be found exactly.

Note also that the accuracy estimates in Theorems 1, 2, and 3 do not give the order of error by δ. In further results this order will be obtained.

2.1. The case of homogenous weight functions

Let T be a cone in a linear space, $T_0=T$, $\mu(\cdot)$ be a homogenous measure of degree d, $|\psi(\cdot)|$ be homogenous function of degree η, $|\varphi_j(\cdot)|$, $j=1,\dots,n$, be homogenous functions of degrees ν, $\psi(t)\ne 0$ and $\sum_{j=1}^n|\varphi_j(t)|\ne 0$ for almost all $t\in T$. Let assume, again, that $1\le q<p,r<\infty$. For $k\in[0,1)$ the function $k^{\frac{1}{p-q}}(1-k)^{-\frac{1}{r-q}}$ increases monotonically from 0 to $+\infty$. Consequently, there exists $k(\cdot)$ such that for almost all $t\in T$

$$\frac{k^{\frac{1}{p-q}}(t)}{(1-k(t))^{\frac{1}{r-q}}}=s_r^{-\frac{1}{r-q}}(t)|\psi(t)|^{\frac{q(p-r)}{(p-q)(r-q)}},\quad s_r(t)=\sum_{j=1}^n|\varphi_j(t)|^r. \tag{23}$$

Set

$$k(t)=\begin{cases}\left(1-|\psi(t)|^{-q}s_q(t)\right)_+, & (p,q,r)\in P_1,\\ \min\left\{1,|\psi(t)|^{-p}\right\}, & (p,q,r)\in P_2.\end{cases}$$

Theorem 4. *Let $(p,q,r)\in P\cup P_1\cup P_2$ and $\nu+d(1/r-1/p)\neq 0$. Assume that for $(p,q,r)\in P\cup P_1$*

$$I_1=\int\limits_T |\psi(t)|^{\frac{qp}{p-q}}k^{\frac{p}{p-q}}(t)\,d\mu(t)<\infty,$$

$$I_{j+1}=\int\limits_T |\psi(t)|^{\frac{qr}{p-q}}|\varphi_j(t)|^r k^{\frac{r}{p-q}}(z)\,d\mu(t)<\infty,\ j=1,\dots,n,$$

and for $(p,q,r)\in P_2$

$$I_1=\int\limits_T \left(s_r^{-1}(t)(|\psi(t)|^p-1)_+\right)^{\frac{p}{r-p}}\,d\mu(t)<\infty,$$

$$I_{j+1}=\int\limits_T |\varphi_j(t)|^r\left(s_r^{-1}(t)(|\psi(t)|^p-1)_+\right)^{\frac{r}{r-p}}\,d\mu(t)<\infty,\ j=1,\dots,n.$$

Moreover, assume that $I_2=\ldots=I_{n+1}$. Then

$$E(p,q,r)=\delta^{\gamma}I_1^{-\gamma/p}I_2^{-(1-\gamma)/r}(I_1+nI_2)^{1/q},\tag{24}$$

where

$$\gamma=\frac{\nu-\eta-d(1/q-1/r)}{\nu+d(1/r-1/p)}.\tag{25}$$

The method

$$\widehat{m}(y)(t)=k(\xi t)\psi(t)y(t),$$

where

$$\xi=\left(\delta I_1^{-1/p}I_2^{1/r}\right)^{\frac{1}{\nu+d(1/r-1/p)}},\tag{26}$$

is optimal.

Proof. 1. Let $(p,q,r)\in P$. Put

$$\widehat{x}(t)=\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{1}{p-q}}k^{\frac{1}{p-q}}(\xi t),$$

where λ_0 will be specified later. We have

$$p\lambda_0\widehat{x}^{p-q}(t)=q|\psi(t)|^qk(\xi t)\tag{27}$$

and

$$rs_r(t)\widehat{x}^{r-q}(t)=rs_r(t)\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{r-q}{p-q}}k^{\frac{r-q}{p-q}}(\xi t).$$

Since $|\psi(\cdot)|$ and $|\varphi_j(\cdot)|$, $j=1,\dots,n$, are homogenous it follows by (23) that

$$k^{\frac{r-q}{p-q}}(\xi t)=\frac{|\psi(\xi t)|^{\frac{q(p-r)}{p-q}}}{s_r(\xi t)}(1-k(\xi t))=\xi^{\eta\frac{q(p-r)}{p-q}-\nu r}\frac{|\psi(t)|^{\frac{q(p-r)}{p-q}}}{s_r(t)}(1-k(\xi t)).$$

Thus,

$$rs_r(t)\widehat{x}^{r-q}(t)=r\left(\frac{q}{p\lambda_0}\right)^{\frac{r-q}{p-q}}\xi^{\eta\frac{q(p-r)}{p-q}-\nu r}|\psi(t)|^q(1-k(\xi t)).$$

Put

$$\lambda=\frac{q}{r}\left(\frac{q}{p\lambda_0}\right)^{-\frac{r-q}{p-q}}\xi^{-\eta\frac{q(p-r)}{p-q}+\nu r}. \tag{28}$$

Then

$$r\lambda s_r(t)\widehat{x}^{r-q}(t)=q|\psi(t)|^q(1-k(\xi t)). \tag{29}$$

Taking the sum of (27) and (29), we obtain

$$p\lambda_0\widehat{x}^{p-q}(t)+r\lambda s_r(t)\widehat{x}^{r-q}(t)=q|\psi(t)|^q.$$

It means that $\widehat{x}(\cdot)$ satisfies (5) for $\lambda_1=\ldots=\lambda_n=\lambda$.

Now we show that for

$$\lambda_0=\frac{q}{p}I_1^{\frac{p-q}{p}}\xi^{-\eta q-d\frac{p-q}{p}}\delta^{q-p} \tag{30}$$

the equalities

$$\int_T\widehat{x}^p(t)\,d\mu(t)=\delta^p,\qquad\int_T|\varphi_j(t)|^r\widehat{x}^r(t)\,d\mu(t)=1,\ \ j=1,\ldots,n,$$

hold. In view of the definition of $\widehat{x}(\cdot)$ we need to check that

$$\int_T\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{p}{p-q}}k^{\frac{p}{p-q}}(\xi t)\,d\mu(t)=\delta^p,$$

$$\int_T|\varphi_j(t)|^r\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{r}{p-q}}k^{\frac{r}{p-q}}(\xi t)\,d\mu(t)=1,\ \ j=1,\ldots,n.$$

Changing $z=\xi t$ and taking into account that functions $|\psi(\cdot)|$, $|\varphi_j(\cdot)|$, $j=1,\ldots,n$, with the measure $\mu(\cdot)$ are homogenous, we obtain

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{p}{p-q}}I_1=\delta^p\xi^{\frac{\eta qp}{p-q}+d},\qquad\left(\frac{q}{p\lambda_0}\right)^{\frac{r}{p-q}}I_{j+1}=\xi^{\frac{\eta qr}{p-q}+\nu r+d},\ \ j=1,\ldots,n.$$

The validity of these equalities immediately follows from the definitions of λ_0 and ξ.

It follows by Theorem 1, (30), (28), and (26) that

$$E^q(p,q,r)=\frac{p\lambda_0\delta^p+nr\lambda}{q}=I_1^{\frac{p-q}{p}}\xi^{-\eta q-d\frac{p-q}{p}}\delta^q$$
$$+n\left(\frac{p\lambda_0}{q}\right)^{\frac{r-q}{p-q}}\xi^{\nu r-\eta\frac{q(p-r)}{p-q}}=\delta^{q\gamma}I_1^{-q\gamma/p}I_2^{-q(1-\gamma)/r}(I_1+nI_2).$$

Moreover, the same theorem states that the method

$$\widehat{m}(y)(t)=q^{-1}p\lambda_0\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t)=k(\xi t)\psi(t)y(t)$$

is optimal.

2. Let $(p,q,r)\in P_1$. We use Theorem 2. Consider the function $\widehat{x}(\cdot)$ defined by (14) with $\lambda_1=\ldots=\lambda_n=\lambda$. Let us find λ_0 and λ from the conditions

$$\int\limits_T \widehat{x}^p(t)\,d\mu(t)=\delta^p,\quad \int\limits_T |\varphi_j(t)|^q\widehat{x}^q(t)\,d\mu(t)=1,\ j=1,\ldots,n.$$

Then we obtain

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{p}{p-q}}\int\limits_T \left(|\psi(t)|^q-\lambda s_q(t)\right)_+^{\frac{p}{p-q}}\,d\mu(t)=\delta^p,$$

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{q}{p-q}}\int\limits_T |\varphi_j(t)|^q\left(|\psi(t)|^q-\lambda s_q(t)\right)_+^{\frac{q}{p-q}}\,d\mu(t)=1,\ j=1,\ldots,n.$$

Put $\lambda=a^{(\eta-\nu)q}$, $a>0$. Changing $t=az$, we obtain

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{p}{p-q}}a^{d+\frac{pq\eta}{p-q}}I_1=\delta^p,\quad \left(\frac{q}{p\lambda_0}\right)^{\frac{q}{p-q}}a^{d+q\nu+\frac{q^2\eta}{p-q}}I_{j+1}=1,\ j=1,\ldots,n.$$

It is easy to check that these equalities are fulfilled for

$$a=(I_1^{1/p}I_2^{-1/q}\delta^{-1})^{\frac{1}{\nu+d(1/q-1/p)}},\quad \lambda_0=\frac{q}{p}I_1I_2^{-1}\delta^{-p}(I_1^{-q/p}I_2\delta^q)^{\frac{\eta-\nu}{\nu+d(1/q-1/p)}}.$$

Substituting these values in (15) and (16) we obtain the statement of the theorem in the case under consideration.

3. Let $(p,q,r)\in P_2$. Here we use Theorem 3. Put $\lambda_1=\ldots=\lambda_n=\lambda$ in the definition of $\widehat{x}(\cdot)$ (see (18)). We find λ_0 and λ from the conditions

$$\int\limits_T \widehat{x}^p(t)\,d\mu(t)=\delta^p,\quad \int\limits_T |\varphi_j(t)|^r\widehat{x}^r(t)\,d\mu(t)=1,\ j=1,\ldots,n.$$

We have

$$\left(\frac{p}{r\lambda}\right)^{\frac{p}{r-p}}\int\limits_T \left(s_r^{-1}(t)(|\psi(t)|^p-\lambda_0)_+\right)^{\frac{p}{r-p}}\,d\mu(t)=\delta^p,$$

$$\left(\frac{p}{r\lambda}\right)^{\frac{r}{r-p}}\int\limits_T |\varphi_j(t)|^r\left(s_r^{-1}(t)(|\psi(t)|^p-\lambda_0)_+\right)^{\frac{r}{r-p}}\,d\mu(t)=1,\ j=1,\ldots,n.$$

Put $\lambda_0=a^{\eta p}$, $a>0$. Changing $t=az$, we obtain

$$\left(\frac{p}{r\lambda}\right)^{\frac{p}{r-p}}a^{d+\frac{p^2\eta}{r-p}-\frac{pr\nu}{r-p}}I_1=\delta^p,$$

$$\left(\frac{p}{r\lambda}\right)^{\frac{r}{r-p}}a^{d+r\nu+\frac{pr\eta}{r-p}-\frac{r^2\nu}{r-p}}I_{j+1}=1,\ j=1,\ldots,n.$$

These equalities are valid for

$$a=(I_1^{1/p}I_2^{-1/r}\delta^{-1})^{\frac{1}{\nu+d(1/r-1/p)}},$$

$$\lambda=\frac{p}{r}I_1^{r/p-1}\delta^{p-r}(I_1^{r/p}I_2^{-1}\delta^{-r})^{\frac{p\eta/r-\nu-d(1/r-1/p)}{\nu+d(1/r-1/p)}}.$$

It remains to substitute these values into (19) and (20). $\square$

Corollary 1. *Assume that conditions of Theorem 4 hold. Then for all $x(\cdot) \neq 0$ such that $x(\cdot) \in L_p(T,\mu)$ and $\varphi_j(\cdot)x(\cdot) \in L_r(T,\mu)$, $j=1,\dots,n$, the sharp inequality*

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le C\|x(\cdot)\|^{\gamma}_{L_p(T,\mu)} \max_{1\le j\le n} \|\varphi_j(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)} \tag{31}$$

holds, where

$$C = I_1^{-\gamma/p} I_2^{-(1-\gamma)/r}(I_1+nI_2)^{1/q}.$$

Proof. Let $x(\cdot) \in L_p(T,\mu)$, $\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} < \infty$, $j=1,\dots,n$ and $x(\cdot) \neq 0$. Put

$$A = \max_{1\le j\le n} \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}.$$

Consider $\widehat{x}(\cdot) = x(\cdot)/A$. Put $\delta = \|\widehat{x}(\cdot)\|_{L_p(T_0,\mu)}$. Then $\|\varphi_j(\cdot)\widehat{x}(\cdot)\|_{L_r(T,\mu)} \le 1$, $j=1,\dots,n$. In view of (22) and Theorem 4 we have

$$\|\psi(\cdot)\widehat{x}(\cdot)\|_{L_q(T,\mu)} \le C\|\widehat{x}(\cdot)\|^{\gamma}_{L_p(T,\mu)}.$$

This implies (31).

If there exists a $\widetilde{C} < C$ for which (31) holds, then

$$E(p,q,r) = \sup_{\substack{\|x(\cdot)\|_{L_p(T,\mu)}\le\delta \\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n}} \|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le \widetilde{C}\delta^{\gamma} < C\delta^{\gamma}.$$

This contradicts with (24). □

Let $|w(\cdot)|$, $|w_0(\cdot)|$ be homogenous functions of degrees θ, θ_0, respectively and $|w_j(\cdot)|$, $j=1,\dots,n$, be homogenous functions of degree θ_1. We assume that $w(t), w_0(t) \neq 0$ and $\sum_{j=1}^n |w_j(t)| \neq 0$ for almost all $t \in T$.

For $(p,q,r) \in P$ we define $\widetilde{k}(\cdot)$ by the equality

$$\frac{\widetilde{k}^{\frac{1}{p-q}}(t)}{(1-\widetilde{k}(t))^{\frac{1}{r-q}}} = \left|\frac{w_0(t)}{w(t)}\right|^{\frac{p}{p-q}} \left(\sum_{j=1}^n \left|\frac{w_j(t)}{w(t)}\right|^r\right)^{-\frac{1}{r-q}}.$$

For $(p,q,r) \in P_1$ set

$$\widetilde{k}(t) = \left(1 - |w(t)|^{-q}\sum_{j=1}^n |w_j(t)|^q\right)_+.$$

Put

$$\widetilde{\theta} = \theta + d/q, \quad \widetilde{\theta}_0 = \theta_0 + d/p, \quad \widetilde{\theta}_1 = \theta_1 + d/r, \quad \widetilde{\gamma} = \frac{\widetilde{\theta}_1 - \widetilde{\theta}}{\widetilde{\theta}_1 - \widetilde{\theta}_0}. \tag{32}$$

Corollary 2. *Let $(p,q,r) \in P \cup P_1 \cup P_2$ and $\widetilde{\theta}_0 \neq \widetilde{\theta}_1$. Assume that for $(p,q,r) \in P \cup P_1$*

$$\widetilde{I}_1 = \int_T \left|\frac{w(t)}{w_0(t)}\right|^{\frac{qp}{p-q}} \widetilde{k}^{\frac{p}{p-q}}(t)\,d\mu(t) < \infty,$$

$$\widetilde{I}_{j+1} = \int_T \frac{|w(t)|^{\frac{qr}{p-q}}}{|w_0(t)|^{\frac{pr}{p-q}}} |w_j(t)|^r \widetilde{k}^{\frac{r}{p-q}}(t)\,d\mu(t) < \infty, \ j=1,\dots,n,$$

and for $(p,q,r)\in P_2$

$$\widetilde{I}_1=\int\limits_T |w_0(t)|^p\left(\frac{(|w(t)|^p-|w_0(t)|^p)_+}{\sum_{k=1}^n |w_k(t)|^r}\right)^{\frac{p}{r-p}}\,d\mu(t)<\infty,$$

$$\widetilde{I}_{j+1}=\int\limits_T |w_j(t)|^r\left(\frac{(|w(t)|^p-|w_0(t)|^p)_+}{\sum_{k=1}^n |w_k(t)|^r}\right)^{\frac{r}{r-p}}\,d\mu(t)<\infty,\ j=1,\dots,n.$$

Moreover, assume that $\widetilde{I}_2=\ldots=\widetilde{I}_{n+1}$. *Then for all* $x(\cdot)\neq 0$ *such that* $w_0(\cdot)x(\cdot)\in L_p(T,\mu)$ *and* $w_j(\cdot)x(\cdot)\in L_r(T,\mu)$, $j=1,\dots,n$, *the sharp inequality*

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le \widetilde{C}\|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu)}\max_{1\le j\le n}\|\omega_j(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)} \tag{33}$$

holds, where

$$\widetilde{C}=\widetilde{I}_1^{-\widetilde{\gamma}/p}\widetilde{I}_2^{-(1-\widetilde{\gamma})/r}(\widetilde{I}_1+n\widetilde{I}_2)^{1/q}.$$

Proof. Set

$$\psi(t)=\frac{w(t)}{w_0(t)},\quad \varphi_j(t)=\frac{w_j(t)}{w_0(t)},\ j=1,\dots,n.$$

Then $|\psi(\cdot)|$ is a homogenous function of degree $\eta=\theta-\theta_0$ and $|\varphi_j(\cdot)|$, $j=1,\dots,n$, are homogenous functions of degrees $\nu=\theta_1-\theta_0$. The quantity γ which was defined by (25) has the following form:

$$\widetilde{\gamma}=\frac{\widetilde{\theta}_1-\widetilde{\theta}}{\widetilde{\theta}_1-\widetilde{\theta}_0}.$$

It follows by Corollary 1 that for all $y(\cdot)\neq 0$ such that $y(\cdot)\in L_p(T,\mu)$ and $\varphi_j(\cdot)y(\cdot)\in L_r(T,\mu)$, $j=1,\dots,n$, the sharp inequality

$$\|\psi(\cdot)y(\cdot)\|_{L_q(T,\mu)}\le \widetilde{C}\|y(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu)}\max_{1\le j\le n}\|\varphi_j(\cdot)y(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)}$$

holds. Substituting $y(\cdot)=w_0(\cdot)x(\cdot)$, we obtain (33). □

2.2. Homogenous weights in $\mathbb{R}^d$

Let T be a cone in $\mathbb{R}^d$, $d\mu(t)=dt$, $|\psi(\cdot)|$ be homogenous function of degree η, $|\varphi_j(\cdot)|$, $j=1,\dots,n$, be homogenous functions of degrees ν, $\psi(t)\neq 0$ and $\sum_{j=1}^n|\varphi_j(t)|\neq 0$ for almost all $t\in T$. Consider the polar transformation

$$\begin{aligned} t_1&=\rho\cos\omega_1,\\ t_2&=\rho\sin\omega_1\cos\omega_2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ t_{d-1}&=\rho\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\cos\omega_{d-1},\\ t_d&=\rho\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1}.\end{aligned}$$

Set $\omega=(\omega_1,\dots,\omega_{d-1})$. For any function $f(\cdot)$ we put

$$\widetilde{f}(\omega)=|f(\cos\omega_1,\dots,\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1})|. \tag{34}$$

Note that if $|f(\cdot)|$ is a homogenous function of degree κ, then $\widetilde{f}(\omega)=\rho^{-\kappa}|f(t)|$. Denote by Ω the range of ω. Since T is a cone, Ω does not depend on ρ. Put

$$J(\omega)=\sin^{d-2}\omega_1\sin^{d-3}\omega_2\ldots\sin\omega_{d-2}.$$

Assume that $\gamma \in (0,1)$, where γ is defined by (25). Put

$$\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}. \tag{35}$$

It is easy to verify that $q^* > q \geq 1$. Moreover,

$$q^* = \frac{pqr(\nu + d(1/r - 1/p))}{\nu r(p-q) - \eta q(p-r)}.$$

Theorem 5. *Let $(p,q,r) \in P \cup P_1 \cup P_2$ and $\gamma \in (0,1)$. Assume that*

$$I = \int_\Omega \frac{\widetilde{\psi}^{q^*}(\omega)}{\widetilde{s}_r^{q^*(1-\gamma)/r}(\omega)} J(\omega)\, d\omega < \infty,$$

and $I'_1 = \ldots = I'_n$, where

$$I'_j = \int_\Omega \frac{\widetilde{\psi}^{q^*}(\omega)\widetilde{\varphi}_j^r(\omega)}{\widetilde{s}_r^{q^*(1-\gamma)/r+1}(\omega)} J(\omega)\, d\omega, \ \ j = 1, \ldots, n.$$

Then

$$E(p,q,r) = K\delta^\gamma, \tag{36}$$

where

$$K = \gamma^{-\frac{\gamma}{p}} \left(\frac{1-\gamma}{n}\right)^{-\frac{1-\gamma}{r}} \left(\frac{B\left(q^*\gamma/p, q^*(1-\gamma)/r\right) I}{|\nu + d(1/r - 1/p)|(\gamma r + (1-\gamma)p)}\right)^{1/q^*},$$

$B(\cdot,\cdot)$ is the Euler beta-function. Moreover, the method

$$\widehat{m}(y)(t) = k\left(\widehat{\xi}^{\frac{1}{\nu + d(1/r - 1/p)}} t\right) \psi(t) y(t),$$

where

$$\widehat{\xi} = \delta\gamma^{-1/p} \left(\frac{1-\gamma}{n}\right)^{1/r} \left(\frac{B\left(q^*\gamma/p, q^*(1-\gamma)/r\right) I}{|\nu + d(1/r - 1/p)|(\gamma r + (1-\gamma)p)}\right)^{1/r - 1/p},$$

is optimal recovery method.

Proof. First of all, we note that $I'_1 + \ldots + I'_n = I$. Consequently, $I'_j = I/n$, $j = 1, \ldots, n$. We will apply Theorem 4.

1. Let $(p,q,r) \in P$. Passing to the polar transformation we obtain

$$\frac{k^{\frac{1}{p-q}}(\rho,\omega)}{(1 - k(\rho,\omega))^{\frac{1}{r-q}}} = \rho^{\frac{\eta q(p-r) - \nu r(p-q)}{(p-q)(r-q)}} \frac{\widetilde{\psi}^{\frac{q(p-r)}{(p-q)(r-q)}}(\omega)}{\widetilde{s}_r^{\frac{1}{r-q}}(\omega)}.$$

Using the same scheme of calculation of I_1 as it was given in [9, Theorem 3], we obtain

$$I_1 = \frac{\gamma}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q}) I,$$

where

$$\widehat{p} = q^* \frac{\gamma}{p}, \quad \widehat{q} = q^* \frac{1-\gamma}{r}.$$

In a similar way we calculate

$$I_{j+1}=\frac{1-\gamma}{pr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I'_j,\ \ j=1,\dots,n.$$

Thus,

$$I_2=\frac{1-\gamma}{npr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I.$$

It remains to substitute these values into (24) and (26).

2. Let $(p,q,r)\in P_1$. Now we use the scheme of calculation of I_1 which was given in [11, Theorem 3]. We obtain

$$\begin{aligned}I_1&=\frac{I}{|\nu-\eta|q}B\left(q^*\gamma/p+2,q^*(1-\gamma)/q\right)\\&=\frac{I}{|\nu-\eta|q}\frac{q^*\gamma/p+1}{q^*\gamma/p+1+q^*(1-\gamma)/q}B\left(q^*\gamma/p+1,q^*(1-\gamma)/q\right).\end{aligned}$$

Since $r=q$ we have

$$\frac{1}{q^*}=\gamma\left(\frac{1}{q}-\frac{1}{p}\right),\quad \gamma=\frac{\nu-\eta}{\nu+d(1/q-1/p)}.$$

Therefore, $q^*\gamma/p+1=q^*\gamma/q$. Hence

$$\begin{aligned}I_1&=\frac{I\gamma}{|\nu-\eta|q}B\left(q^*\gamma/p+1,q^*(1-\gamma)/q\right)\\&=\frac{I\gamma}{|\nu-\eta|q}\frac{q^*\gamma/p}{q^*\gamma/p+q^*(1-\gamma)/q}B\left(q^*\gamma/p,q^*(1-\gamma)/q\right)\\&=\frac{\gamma}{pr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I.\end{aligned}$$

By the similar way we get

$$\begin{aligned}I_{j+1}&=\frac{I'_j}{|\nu-\eta|q}B\left(q^*\gamma/p+1,q^*(1-\gamma)/q+1\right)\\&=\frac{I'_j}{|\nu-\eta|q}\frac{q^*\gamma/p}{q^*\gamma/p+q^*(1-\gamma)/q+1}B\left(q^*\gamma/p,q^*(1-\gamma)/q+1\right)\\&=\frac{I'_j\gamma B\left(q^*\gamma/p,q^*(1-\gamma)/q+1\right)}{|\nu-\eta|p}=\frac{(1-\gamma)B(\widehat{p},\widehat{q})I}{npr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}.\end{aligned}$$

Thus, we obtain the same formulas for I_1 and I_2 as in the first case.

3. Let $(p,q,r)\in P_2$. Here we use the scheme of calculation of J_1 and J_2 which was given in [11, Theorem 3]. We obtain

$$\begin{aligned}I_1&=\frac{I}{|\eta|p}B\left(q^*\gamma/p+1,q^*(1-\gamma)/r+1\right),\\I_{j+1}&=\frac{I'_j}{|\eta|p}B\left(q^*\gamma/p,q^*(1-\gamma)/r+2\right),\ \ j=1,\dots,n.\end{aligned}$$

Since $q=p$ we have

$$\frac{1}{q^*}=(1-\gamma)\left(\frac{1}{p}-\frac{1}{r}\right),\quad 1-\gamma=\frac{\eta}{\nu+d(1/r-1/p)}.$$

Therefore, $q^*(1-\gamma)/r+1 = q^*(1-\gamma)/p$. Hence

$$\begin{aligned}I_1 &= \frac{I}{|\eta|p}\frac{q^*\gamma/p}{q^*\gamma/p+q^*(1-\gamma)/r+1}B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right)\\ &= \frac{I\gamma B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right)}{|\eta|p} = \frac{I\gamma}{|\eta|p}\frac{q^*(1-\gamma)/rB\left(q^*\gamma/p, q^*(1-\gamma)/r\right)}{q^*\gamma/p+q^*(1-\gamma)/r}\\ &= \frac{\gamma B(\widehat{p},\widehat{q})I}{pr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}.\end{aligned}$$

For I_{j+1}, $j=1,\dots,n$, we have

$$\begin{aligned}I_{j+1} &= \frac{I'_j}{|\eta|p}\frac{(q^*(1-\gamma)/r+1)B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right)}{q^*\gamma/p+q^*(1-\gamma)/r+1}\\ &= \frac{I'_j(1-\gamma)B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right)}{|\eta|p}\\ &= \frac{(1-\gamma)B(\widehat{p},\widehat{q})I}{npr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}.\end{aligned}$$

Again we obtain the same formulas for I_1 and I_2 as in the previous cases. □

For $n=1$ Theorem 5 was proved in [11]. Analogously to Corollary 1 we obtain

Corollary 3. *Assume that conditions of Theorem 5 hold. Then for all $x(\cdot)$ such that $x(\cdot)\in L_p(T,\mu)$ and $\varphi_j(\cdot)x(\cdot)\in L_r(T,\mu)$, $j=1,\dots,n$, the sharp inequality*

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le K\|x(\cdot)\|^{\gamma}_{L_p(T,\mu)}\max_{1\le j\le n}\|\varphi_j(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)}$$

holds.

Let $|w(\cdot)|$, $|w_0(\cdot)|$ be homogenous functions of degrees θ, θ_0, respectively and $|w_j(\cdot)|$, $j=1,\dots,n$, be homogenous functions of degree θ_1. We assume that $w(t), w_0(t)\ne 0$ and $\sum_{j=1}^n |w_j(t)|\ne 0$ for almost all $t\in T$. Define $\widetilde{w}(\cdot)$, $\widetilde{w}_0(\cdot)$, $\widetilde{w}_1(\cdot)$ by (34). Similar to Corollary 2 we obtain

Corollary 4. *Let $(p,q,r)\in P\cup P_1\cup P_2$ and $\widetilde{\gamma}\in(0,1)$ where $\widetilde{\gamma}$ is defined by (32). Assume that*

$$\widetilde{I} = \int_\Omega \frac{\widetilde{w}^{\widetilde{q}}(\omega)}{\widetilde{w}_0^{\widetilde{q}\widetilde{\gamma}}(\omega)\left(\sum_{k=1}^n \widetilde{w}_k^r(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r}}J(\omega)\,d\omega<\infty,$$

where

$$\frac{1}{\widetilde{q}} = \frac{1}{q}-\frac{\widetilde{\gamma}}{p}-\frac{1-\widetilde{\gamma}}{r},$$

and $\widetilde{I}'_1=\ldots=\widetilde{I}'_n$, where

$$\widetilde{I}'_j = \int_\Omega \frac{\widetilde{w}^{\widetilde{q}}(\omega)\widetilde{w}_j^r(\omega)}{\widetilde{w}_0^{\widetilde{q}\widetilde{\gamma}}(\omega)\left(\sum_{k=1}^n \widetilde{w}_k^r(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r+1}}J(\omega)\,d\omega,\ \ j=1,\dots,n.$$

Then for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot)\in L_p(T,\mu)$ and $w_j(\cdot)x(\cdot)\in L_r(T,\mu)$, $j=1,\dots,n$, the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le \widetilde{K}\|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu))}\max_{1\le j\le n}\|w_j(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)}$$

holds, where

$$\widetilde{K}=\widetilde{\gamma}^{-\frac{\widetilde{\gamma}}{p}}\left(\frac{1-\widetilde{\gamma}}{n}\right)^{-\frac{1-\widetilde{\gamma}}{r}}\left(\frac{B\left(\widetilde{q}\widetilde{\gamma}/p,\widetilde{q}(1-\widetilde{\gamma})/r\right)\widetilde{I}}{|\widetilde{\theta}_1-\widetilde{\theta}_0|(\widetilde{\gamma}r+(1-\widetilde{\gamma})p)}\right)^{1/\widetilde{q}}. \tag{37}$$

The statement of Corollary 4 for $(p,q,r)\in P$ and $n=1$ was proved in [2].

We give an example of weights for which conditions of Corollary 4 hold. Let $T=\mathbb{R}^d_+$, $\theta_1>0$,

$$w(t)=(t_1^2+\ldots+t_d^2)^{\theta/2},\quad w_0(t)=(t_1^2+\ldots+t_d^2)^{\theta_0/2},\quad w_j(t)=t_j^{\theta_1},\ j=1,\ldots,d. \tag{38}$$

The condition $0<\widetilde{\gamma}<1$ is equivalent to inequalities $\widetilde{\theta}_1>\widetilde{\theta}>\widetilde{\theta}_0$ or $\widetilde{\theta}_1<\widetilde{\theta}<\widetilde{\theta}_0$. Therefore, we assume that for θ and θ_0 inequalities $\theta_1+d(1/r-1/q)>\theta>\theta_0+d(1/p-1/q)$ or $\theta_1+d(1/r-1/q)<\theta<\theta_0+d(1/p-1/q)$ hold.

It is easy to check that $\widetilde{w}(\cdot)=\widetilde{w}_0(\cdot)=1$ and $\widetilde{w}_j(\omega)=\widetilde{t}_j^{\,\theta_1}(\omega)$, $j=1,\ldots,d$, where

$$\begin{aligned}\widetilde{t}_1(\omega)&=\cos\omega_1,\\ \widetilde{t}_2(\omega)&=\sin\omega_1\cos\omega_2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \widetilde{t}_{d-1}(\omega)&=\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\cos\omega_{d-1},\\ \widetilde{t}_d(\omega)&=\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1}.\end{aligned}$$

Note that

$$\sum_{k=1}^d\widetilde{t}_k^{\,2}(\omega)=1.$$

For $\widetilde{I}$ we have

$$\widetilde{I}=\int\limits_{\Pi_+^{d-1}}\frac{J(\omega)\,d\omega}{\left(\sum_{k=1}^d\widetilde{t}_k^{\,r\theta_1}(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r}},\quad \Pi_+^{d-1}=[0,\pi/2]^{d-1}. \tag{39}$$

If $r\theta_1\le 2$, then

$$\sum_{k=1}^d\widetilde{t}_k^{\,r\theta_1}(\omega)\ge\sum_{k=1}^d\widetilde{t}_k^{\,2}(\omega)=1. \tag{40}$$

For $r\theta_1>2$ by Hölder's inequality

$$1=\sum_{k=1}^d\widetilde{t}_k^{\,2}(\omega)\le\left(\sum_{k=1}^d\widetilde{t}_k^{\,r\theta_1}(\omega)\right)^{\frac{2}{r\theta_1}}d^{1-\frac{2}{r\theta_1}}.$$

Thus,

$$\sum_{k=1}^d\widetilde{t}_k^{\,r\theta_1}(\omega)\ge d^{1-\frac{r\theta_1}{2}}. \tag{41}$$

It follows by (40) and (41) that $\widetilde{I}<\infty$.

For $\widetilde{I}'_j$ we have

$$\widetilde{I}'_j=\int\limits_{\Pi_+^{d-1}}\frac{\widetilde{t}_j^{\,r\theta_1}J(\omega)\,d\omega}{\left(\sum_{k=1}^d\widetilde{t}_k^{\,r\theta_1}(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r+1}},\ j=1,\ldots,d.$$

Consider the integrals

$$L_j = \int\limits_{\mathbb{R}^d_+\cap\mathbb{B}^d} \frac{\left(\sum_{k=1}^d t_k^2\right)^{\theta_1\widetilde{q}(1-\widetilde{\gamma})/2} t_j^{r\theta_1}}{\left(\sum_{k=1}^d t_k^{r\theta_1}\right)^{\widetilde{q}(1-\widetilde{\gamma})/r+1}}\,dt, \ \ j=1,\dots,d,$$

where $\mathbb{B}^d$ is the unit ball in $\mathbb{R}^d$. If we change variables in L_j changing places variables t_j and t_k, then L_j passes to L_k. Therefore, $L_1=\ldots=L_d$. Passing to the polar transformation we obtain that $L_j=\widetilde{I}'_j/d$, $j=1,\dots,d$. Consequently, $\widetilde{I}'_1=\ldots=\widetilde{I}'_d$.

Thus, we obtain

Corollary 5. *Let $(p,q,r)\in P\cup P_1\cup P_2$, $\theta_1>0$, θ and θ_0 be such that $\theta_1+d(1/r-1/q)>\theta>\theta_0+d(1/p-1/q)$ or $\theta_1+d(1/r-1/q)<\theta<\theta_0+d(1/p-1/q)$. Then for weights (38) and all $x(\cdot)$ for which $w_0(\cdot)x(\cdot)\in L_p(\mathbb{R}^d_+)$ and $w_j(\cdot)x(\cdot)\in L_r(\mathbb{R}^d_+)$, $j=1,\dots,d$, the sharp inequality*

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d_+)}\le \widetilde{K}\|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(\mathbb{R}^d_+)}\max_{1\le j\le d}\|w_j(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(\mathbb{R}^d_+)}$$

holds, where $\widetilde{K}$ is defined by (37) in which the value $\widetilde{I}$ is defined by (39).

We give one more example.

Corollary 6. *Let $(p,q,r)\in P\cup P_1\cup P_2$, weights $w(\cdot)$, $w_0(\cdot)$, $w_j(\cdot)$, $j=1,\dots,d$, be defined by (38) for $\theta=d(1-1/q)$, $\theta_0=d-(\lambda+d)/p$, $\theta_1=d+(\mu-d)/r$, where $\lambda,\mu>0$. Put*

$$\alpha=\frac{\mu}{p\mu+r\lambda},\quad \beta=\frac{\lambda}{p\mu+r\lambda}.$$

Then for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot)\in L_p(\mathbb{R}^d_+)$ and $w_j(\cdot)x(\cdot)\in L_r(\mathbb{R}^d_+)$, $j=1,\dots,d$, the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d_+)}\le C\|w_0(\cdot)x(\cdot)\|^{p\alpha}_{L_p(\mathbb{R}^d_+)}\max_{1\le j\le d}\|w_j(\cdot)x(\cdot)\|^{r\beta}_{L_r(\mathbb{R}^d_+)}$$

holds, where

$$C=\frac{d^\beta}{(p\alpha)^\alpha(r\beta)^\beta}\left(\frac{I}{\lambda+\mu}B\left(\frac{\alpha}{1/q-\alpha-\beta},\frac{\beta}{1/q-\alpha-\beta}\right)\right)^{1/q-\alpha-\beta},$$

and

$$I=\int\limits_{\Pi^{d-1}_+}\frac{J(\omega)\,d\omega}{\left(\sum_{k=1}^d\widetilde{t}_k^{\,r(d-1)+\mu}(\omega)\right)^{\frac{\beta}{1/q-\alpha-\beta}}}.$$

For $d=1$, $q=1$, and $(p,1,r)\in P$ the statement of Corollary 6 was proved in [5].

3. Recovery of differential operators from a noisy Fourier transform

Let T be a cone in $\mathbb{R}^d$, $d\mu(t)=dt$, $|\psi(\cdot)|$ be homogenous function of degree η, $|\varphi_j(\cdot)|$, $j=1,\dots,n$, be homogenous functions of degrees ν, $\psi(t)\ne0$ and $\sum_{j=1}^n|\varphi_j(t)|\ne0$ for almost all $t\in T$.

Let S be the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$, S' be the corresponding space of distributions, and let $F\colon S'\to S'$ be the Fourier transform. Set

$$X_p=\left\{x(\cdot)\in S':\varphi_j(\cdot)Fx(\cdot)\in L_2(\mathbb{R}^d),\ j=1,\dots,n,\ Fx(\cdot)\in L_p(\mathbb{R}^d)\right\}.$$

We define operators D_j, $j=1,\dots,n$, as follows

$$D_jx(\cdot)=F^{-1}(\varphi_j(\cdot)Fx(\cdot))(\cdot),\ j=1,\dots,n.$$

Put

$$\Lambda x(\cdot)=F^{-1}(\psi(\cdot)Fx(\cdot))(\cdot). \tag{42}$$

Consider the problem of the optimal recovery of values of the operator Λ on the class

$$W_p^{\mathcal{D}}=\left\{x(\cdot)\in X_p:\|D_jx(\cdot)\|_{L_2(\mathbb{R}^d)}\le 1,\ j=1,\dots,n\right\},\quad \mathcal{D}=(D_1,\dots,D_n),$$

from the noisy Fourier transform of the function $x(\cdot)$. We assume that for each $x(\cdot)\in W_p^{\mathcal{D}}$ one knows a function $y(\cdot)\in L_p(\mathbb{R}^d)$ such that $\|Fx(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta$, $\delta>0$. It is required to recover the function $\Lambda x(\cdot)$ from $y(\cdot)$. Assume that $\Lambda x(\cdot)\in L_q(\mathbb{R}^d)$ for all $x(\cdot)\in X_p$. As recovery methods we consider all possible mappings $m\colon L_p(\mathbb{R}^d)\to L_q(\mathbb{R}^d)$. The error of a method m is defined by

$$e_{pq}(\Lambda,\mathcal{D},m)=\sup_{\substack{x(\cdot)\in W_p^{\mathcal{D}},\ y(\cdot)\in L_p(\mathbb{R}^d)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\|\Lambda x(\cdot)-m(y)(\cdot)\|_{L_q(\mathbb{R}^d)}.$$

The quantity

$$E_{pq}(\Lambda,\mathcal{D})=\inf_{m\colon L_p(\mathbb{R}^d)\to L_q(\mathbb{R}^d)}e_{pq}(\Lambda,\mathcal{D},m) \tag{43}$$

is called the error of optimal recovery, and the method on which the infimum is attained, an optimal method.

3.1. Recovery in the metric $L_2(\mathbb{R}^d)$

By Plancherel's theorem,

$$\|\Lambda x(\cdot)-m(y)(\cdot)\|_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^{d/2}}\|\psi(\cdot)Fx(\cdot)-F(m(y))(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Moreover,

$$\|D_jx(\cdot)\|_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^{d/2}}\|\varphi_j(\cdot)Fx(\cdot)\|_{L_2(\mathbb{R}^d)},\ j=1,\dots,n.$$

Put

$$W=\left\{z(\cdot)\in L_p(\mathbb{R}^d):\left\|\frac{1}{(2\pi)^{d/2}}\varphi_j(\cdot)z(\cdot)\right\|_{L_2(\mathbb{R}^d)}\le 1,\ j=1,\dots,n.\right\}.$$

If $x(\cdot)\in W_p^{\mathcal{D}}$, $y(\cdot)\in L_p(\mathbb{R}^d)$, and $\|Fx(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta$, then $z(\cdot)=Fx(\cdot)\in W$ and $\|z(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta$. On the other hand, if $z(\cdot)\in W$ and $\|z(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta$, then $x(\cdot)=F^{-1}z(\cdot)\in W_p^{\mathcal{D}}$ and $\|Fx(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta$. Therefore,

$$\sup_{\substack{x(\cdot)\in W_p^{\mathcal{D}},\ y(\cdot)\in L_p(\mathbb{R}^d)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\left\|\frac{1}{(2\pi)^{d/2}}\psi(\cdot)Fx(\cdot)-\frac{1}{(2\pi)^{d/2}}F(m(y))(\cdot)\right\|_{L_2(\mathbb{R}^d)}$$
$$=\sup_{\substack{z(\cdot)\in W,\ y(\cdot)\in L_p(\mathbb{R}^d)\\ \|z(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\left\|\frac{1}{(2\pi)^{d/2}}\psi(\cdot)z(\cdot)-\frac{1}{(2\pi)^{d/2}}F(m(y))(\cdot)\right\|_{L_2(\mathbb{R}^d)}.$$

Thus,

$$e_{p2}(\Lambda,\mathcal{D},m)=\sup_{\substack{z(\cdot)\in W,\ y(\cdot)\in L_p(\mathbb{R}^d)\\ \|z(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\left\|\frac{1}{(2\pi)^{d/2}}\psi(\cdot)z(\cdot)-\frac{1}{(2\pi)^{d/2}}F(m(y))(\cdot)\right\|_{L_2(\mathbb{R}^d)}.$$

Consequently,

$$E_{p2}(\Lambda,\mathcal{D})\ge\inf_{\widetilde{m}\colon L_p(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\sup_{\substack{z(\cdot)\in W,\ y(\cdot)\in L_p(\mathbb{R}^d)\\ \|z(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\left\|\frac{1}{(2\pi)^{d/2}}\psi(\cdot)z(\cdot)-\widetilde{m}(y)(\cdot)\right\|_{L_2(\mathbb{R}^d)}=\widetilde{E}(p,2,2),\quad(44)$$

where $\widetilde{E}(p,2,2)$ is the value of problem (3) for $q=r=2$ with $\varphi_j(\cdot)$ replaced by $(2\pi)^{-d/2}\varphi_j(\cdot)$, $j=1,\dots,n$, and $\psi(\cdot)$ replaced by $(2\pi)^{-d/2}\psi(\cdot)$.

Suppose that $\widehat{m}\colon L_p(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$ is an optimal method for $\widetilde{E}(p,2,2)$. Then

$$\widetilde{E}(p,2,2)=\sup_{\substack{z(\cdot)\in W,\ y(\cdot)\in L_p(\mathbb{R}^d)\\ \|z(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\left\|\frac{1}{(2\pi)^{d/2}}\psi(\cdot)z(\cdot)-\widehat{m}(y)(\cdot)\right\|_{L_2(\mathbb{R}^d)}$$
$$=\sup_{\substack{x(\cdot)\in W_p^{\mathcal{D}},\ y(\cdot)\in L_p(\mathbb{R}^d)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\|\Lambda x(\cdot)-(2\pi)^{d/2}F^{-1}(\widehat{m}(y))(\cdot)\|_{L_2(\mathbb{R}^d)}\ge E_{p2}(\Lambda,\mathcal{D}).\quad(45)$$

It follows from (44) and (45) that $E_{p2}(\Lambda,\mathcal{D})=\widetilde{E}(p,2,2)$. Moreover, if $\widehat{m}$ is an optimal method for $\widetilde{E}(p,2,2)$, then $(2\pi)^{d/2}F^{-1}(\widehat{m}(y))$ is an optimal method for $E_{p2}(\Lambda,\mathcal{D})$.

For $q=r=2$ we denote by $\widehat{\gamma}$ and $\widehat{q}^*$ the values γ and q^*, which were defined by (25) and (35):

$$\widehat{\gamma}=\frac{\nu-\eta}{\nu+d(1/2-1/p)},\quad \widehat{q}^*=\frac{1}{\widehat{\gamma}(1/2-1/p)}.$$

Set

$$C_p(\nu,\eta)=\widehat{\gamma}^{-\frac{\widehat{\gamma}}{p}}\left(\frac{1-\widehat{\gamma}}{n}\right)^{-\frac{1-\widehat{\gamma}}{2}}\left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1,\widehat{q}^*(1-\widehat{\gamma})/2\right)}{2|\nu-\eta|}\right)^{1/\widehat{q}^*}.$$

Theorem 6. *Let* $2<p\le\infty$, $\widehat{\gamma}\in(0,1)$. *Assume that*

$$I=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}^*}(\omega)}{\widetilde{s}_2^{\widehat{q}^*(1-\widehat{\gamma})/2}(\omega)}J(\omega)\,d\omega<\infty,\quad \Pi^{d-1}=[0,\pi]^{d-2}\times[0,2\pi]\quad(46)$$

and $I_1'=\ldots=I_n'$, *where*

$$I_j'=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}^*}(\omega)\widetilde{\varphi}_j^2(\omega)}{\widetilde{s}_2^{\widehat{q}^*(1-\widehat{\gamma})/2+1}(\omega)}J(\omega)\,d\omega,\ j=1,\dots,n.\quad(47)$$

Then

$$E_{p2}(\Lambda,\mathcal{D})=\frac{1}{(2\pi)^{d\widehat{\gamma}/2}}C_p(\nu,\eta)I^{1/\widehat{q}^*}\delta^{\widehat{\gamma}}.\quad(48)$$

The method

$$\widehat{m}(y)(t)=F^{-1}\left(\left(1-\beta\frac{s_2(t)}{|\psi(t)|^2}\right)_+\psi(t)y(t)\right), \tag{49}$$

where

$$\beta=\frac{1-\widehat{\gamma}}{n(2\pi)^{d\widehat{\gamma}}}C_p^2(\nu,\eta)\left(\delta I^{1/2-1/p}\right)^{2\widehat{\gamma}},$$

is optimal.

Moreover, the sharp inequality

$$\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\frac{C_p(\nu,\eta)I^{1/\widehat{q}^*}}{(2\pi)^{d\widehat{\gamma}/2}}\|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\widehat{\gamma}}\max_{1\le j\le n}\|D_jx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\widehat{\gamma}} \tag{50}$$

holds.

Proof. Let $2<p<\infty$. By Theorem 5 we have

$$E_{p2}(\Lambda,\mathcal{D})=\frac{1}{(2\pi)^{d\widehat{\gamma}/2}}K\delta^{\widehat{\gamma}},$$

where

$$K=\widehat{\gamma}^{-\frac{\widehat{\gamma}}{p}}\left(\frac{1-\widehat{\gamma}}{n}\right)^{-\frac{1-\widehat{\gamma}}{2}}\left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p,\widehat{q}^*(1-\widehat{\gamma})/2\right)I}{|\nu+d(1/2-1/p)|(2\widehat{\gamma}+(1-\widehat{\gamma})p)}\right)^{1/\widehat{q}^*}.$$

From the properties of the beta-function we find that

$$\begin{aligned}&\frac{B\left(\widehat{q}^*\widehat{\gamma}/p,\widehat{q}^*(1-\widehat{\gamma})/2\right)}{|\nu+d(1/2-1/p)|(2\widehat{\gamma}+(1-\widehat{\gamma})p)}\\&\qquad=\frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1,\widehat{q}^*(1-\widehat{\gamma})/2\right)\left(\widehat{q}^*\widehat{\gamma}/p+\widehat{q}^*(1-\widehat{\gamma})/2\right)}{|\nu+d(1/2-1/p)|(2\widehat{\gamma}+(1-\widehat{\gamma})p)\widehat{q}^*\widehat{\gamma}/p}\\&\qquad=\frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1,\widehat{q}^*(1-\widehat{\gamma})/2\right)}{2|\nu-\eta|}.\end{aligned} \tag{51}$$

Thus, equality (48) holds.

It follows by Theorem 5 that the method

$$\widehat{m}(y)(t)=\left(1-\frac{\widehat{\xi}^{2\widehat{\gamma}}s_2(t)}{|\psi(t)|^2}\right)_+(2\pi)^{-d/2}\psi(t)y(t),$$

where

$$\widehat{\xi}=\frac{\delta}{(2\pi)^{d/2}}\widehat{\gamma}^{-1/p}\left(\frac{1-\widehat{\gamma}}{n}\right)^{1/2}\left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p,\widehat{q}^*(1-\widehat{\gamma})/2\right)I}{|\nu+d(1/2-1/p)|(2\widehat{\gamma}+(1-\widehat{\gamma})p)}\right)^{1/2-1/p},$$

is optimal. In view of (51) we obtain

$$\begin{aligned}\widehat{\xi}^{2\widehat{\gamma}}&=\frac{\delta^{2\widehat{\gamma}}\widehat{\gamma}^{-2\widehat{\gamma}/p}}{(2\pi)^{d\widehat{\gamma}}}\left(\frac{1-\widehat{\gamma}}{n}\right)^{\widehat{\gamma}}\left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1,\widehat{q}^*(1-\widehat{\gamma})/2\right)I}{2|\nu-\eta|}\right)^{2\widehat{\gamma}(1/2-1/p)}\\&=\frac{1-\widehat{\gamma}}{n(2\pi)^{d\widehat{\gamma}}}C_p^2(\nu,\eta)\left(\delta I^{1/2-1/p}\right)^{2\widehat{\gamma}}.\end{aligned}$$

Inequality (50) follows from Corollary 3. Consider the case $p=\infty$. It follows by Lemma 1 that

$$E_{\infty 2}(\Lambda,\mathcal{D})\ge \sup_{\substack{x(\cdot)\in W_\infty^{\mathcal{D}}\\ \|Fx(\cdot)\|_{L_\infty(\mathbb{R}^d)}\le\delta}} \|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}. \tag{52}$$

Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi)=\begin{cases}\delta, & |\psi(\xi)|>\lambda\sqrt{s_2(\xi)},\\ 0, & |\psi(\xi)|\le\lambda\sqrt{s_2(\xi)}.\end{cases}$$

We show that $\lambda>0$ may be selected from the condition

$$\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\varphi_j(\xi)|^2|F\widehat{x}(\xi)|^2\,d\xi=1,\ \ j=1,\dots,n.$$

Thus, $\lambda>0$ should be chosen from the condition

$$\delta^2\int\limits_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}|\varphi_j(\xi)|^2\,d\xi=(2\pi)^d.$$

Passing to the polar transformation for $\nu>\eta$ we obtain

$$\delta^2\int\limits_{\Pi_{d-1}}\widetilde{\varphi}_j^2(\omega)J(\omega)\,d\omega\int\limits_0^{\Phi_1(\omega)}\rho^{2\nu+d-1}\,d\rho=(2\pi)^d,\ \ \Phi_1(\omega)=\left(\frac{\widetilde{\psi}(\omega)}{\lambda\sqrt{\widetilde{s}_2(\xi)}}\right)^{\frac{1}{\nu-\eta}}.$$

If $\nu<\eta$, then $2\nu+d<0$ (since $\widehat{\gamma}\in(0,1)$) and we have

$$\delta^2\int\limits_{\Pi_{d-1}}\widetilde{\varphi}_j^2(\omega)J(\omega)\,d\omega\int\limits_{\Phi_1(\omega)}^{+\infty}\rho^{2\nu+d-1}\,d\rho=(2\pi)^d.$$

Hence

$$\frac{\delta^2}{|2\nu+d|}\lambda^{-\frac{2\nu+d}{\nu-\eta}}I_j'=(2\pi)^d.$$

As already noted, it follows from the equality $I_1'+\ldots+I_n'=I$ that $I_j'=I/n$, $j=1,\dots,n$. Consequently,

$$\lambda=\left(\frac{\delta^2 I}{(2\pi)^d n|2\nu+d|}\right)^{\frac{\nu-\eta}{2\nu+d}}.$$

It is easily checked that

$$C_\infty^2(\nu,\eta)=\frac{1}{|2\eta+d|}(n|2\nu+d|)^{\frac{\eta+d/2}{\nu+d/2}}.$$

As a result, $\lambda^2=\beta$. In view of (52), using calculations similar to those that were above, we obtain

$$E_{\infty 2}^2(\Lambda,\mathcal{D})\ge\|\Lambda\widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)}^2=\frac{\delta^2}{(2\pi)^d}\int\limits_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}|\psi(\xi)|^2\,d\xi$$
$$=\frac{\delta^2}{|2\eta+d|(2\pi)^d}\lambda^{-\frac{2\eta+d}{\nu-\eta}}I=\frac{1}{(2\pi)^{d\widehat{\gamma}}}C_\infty^2(\nu,\eta)I^{2/\widehat{q}^*}\delta^{2\widehat{\gamma}}. \tag{53}$$

We estimate the error of the method (49). Put

$$a(\xi)=\left(1-\beta\frac{s_2(\xi)}{|\psi(\xi)|^2}\right)_+.$$

Taking the Fourier transform we obtain

$$\|\Lambda x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\psi(\xi)|^2\,|Fx(\xi)-a(\xi)y(\xi)|^2\,d\xi.$$

We set $z(\cdot)=Fx(\cdot)-y(\cdot)$ and note that

$$\|z(\cdot)\|_{L_\infty(\mathbb{R}^d)}\le\delta,\quad \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\varphi_j(\xi)|^2|Fx(\xi)|^2\,d\xi\le 1,\ j=1,\dots,n.$$

Hence

$$\|\Lambda x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\psi(\xi)|^2\,|(1-a(\xi))\,Fx(\xi)+a(\xi)z(\xi)|^2\,d\xi.$$

The integrand can be written as

$$\left|\frac{|\psi(\xi)|(1-a(\xi))\sqrt{\beta s_2(\xi)}Fx(\xi)}{\sqrt{\beta s_2(\xi)}}+\sqrt{a(\xi)}\sqrt{a(\xi)}|\psi(\xi)|z(\xi)\right|^2.$$

Using the Cauchy-Bunyakovskii-Schwarz inequality

$$|ab+cd|^2\le(|a|^2+|c|^2)(|b|^2+|d|^2)$$

we obtain the estimate

$$\begin{aligned}&\|\Lambda x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}\\&\qquad\le\operatorname*{vraisup}_{\xi\in\mathbb{R}^d}S(\xi)\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\left(\beta s_2(\xi)|Fx(\xi)|^2+a(\xi)|\psi(\xi)|^2|z(\xi)|^2\right)d\xi,\end{aligned}$$

where

$$S(\xi)=\frac{|\psi(\xi)|^2|(1-a(\xi))^2}{\beta s_2(\xi)}+a(\xi).$$

If $|\psi(\xi)|^2\le\beta s_2(\xi)$, then $a(\xi)=0$ and $S(\xi)\le 1$. If $|\psi(\xi)|^2>\beta s_2(\xi)$, then $S(\xi)=1$. So we have

$$\begin{aligned}e^2_{\infty 2}(\Lambda,\mathcal{D},\widehat{m})&\le\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\left(\beta s_2(\xi)|Fx(\xi)|^2+a(\xi)|\psi(\xi)|^2|z(\xi)|^2\right)d\xi\\&\le n\beta+\frac{\delta^2}{(2\pi)^d}\int\limits_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}\left(|\psi(\xi)|^2-\beta s_2(\xi)\right)d\xi\\&=n\beta+\frac{\delta^2}{(2\pi)^d}\int\limits_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}|\psi(\xi)|^2\,d\xi-\beta\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}s_2(\xi)|F\widehat{x}(\xi)|^2\,d\xi\\&=\frac{\delta^2}{(2\pi)^d}\int\limits_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}|\psi(\xi)|^2\,d\xi\le E^2_{\infty 2}(\Lambda,\mathcal{D}).\end{aligned}$$

It follows that the method $\widehat{m}(y)(\cdot)$ is optimal. Moreover, by (53) we have

$$E_{\infty 2}^2(\Lambda, \mathcal{D}) = \frac{\delta^2}{(2\pi)^d} \int\limits_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}} |\psi(\xi)|^2\, d\xi = \frac{1}{(2\pi)^{d\widehat{\gamma}}} C_\infty^2(n,k) I^{2/\widehat{q}^*} \delta^{2\widehat{\gamma}}.$$

Similar to the proof of Corollary 1 we prove that for $p=\infty$ inequality (50) is sharp. □

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$. We define the operator D^α (the derivative of order α) by

$$D^\alpha x(\cdot) = F^{-1}((i\xi)^\alpha Fx(\xi))(\cdot),$$

where $(i\xi)^\alpha = (i\xi_1)^{\alpha_1} \dots (i\xi_d)^{\alpha_d}$.

Consider problem (43) for $D_j = D^{\nu e_j}$, $j = 1, \dots, d$, where e_j, $j = 1 \dots, d$, is a standard basis in $\mathbb{R}^d$, and Λ defined by (42). Assume that $\psi(\cdot)$ has the following symmetry property

$$\psi(\dots, \xi_j, \dots, \xi_m, \dots) = \psi(\dots, \xi_m, \dots, \xi_j, \dots), \quad 1 \le j, m \le d.$$

Moreover, we assume that $\widetilde{\psi}(\cdot)$ is continuous function on Π^{d-1}.

In this case for (46) and (47) we have

$$I = \int\limits_{\Pi^{d-1}} \frac{\widetilde{\psi}^{\widehat{q}^*}(\omega) J(\omega)\, d\omega}{\left(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\right)^{\widehat{q}^*(1-\widehat{\gamma})/2}}, \tag{54}$$

$$I_j' = \int\limits_{\Pi^{d-1}} \frac{\widetilde{\psi}^{\widehat{q}^*}(\omega) \widetilde{t}_j^{2\nu}(\omega) J(\omega)\, d\omega}{\left(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\right)^{\widehat{q}^*(1-\widehat{\gamma})/2+1}}, \quad j = 1, \dots, d.$$

Similar to how it was done for weights (38) we prove that $I < \infty$ and $I_1' = \dots = I_d'$. Thus, from Theorem 6 we obtain

Corollary 7. *Let $2 < p \le \infty$ and $\nu > \eta \ge 0$. Then*

$$E_{p2}(\Lambda, (D^{\nu e_1}, \dots, D^{\nu e_d})) = \frac{1}{(2\pi)^{d\widehat{\gamma}/2}} C_p(\nu, \eta) I^{1/\widehat{q}^*} \delta^{\widehat{\gamma}},$$

where I is defined by (54). The method

$$\widehat{m}(y)(t) = F^{-1}\left(\left(1 - \beta \frac{\sum_{j=1}^d |t_j|^{2\nu}}{|\psi(t)|^2}\right)_+ \psi(t) y(t)\right),$$

where

$$\beta = \frac{1-\widehat{\gamma}}{d(2\pi)^{d\widehat{\gamma}}} C_p^2(\nu, \eta) \left(\delta I^{1/2-1/p}\right)^{2\widehat{\gamma}},$$

is optimal.

The sharp inequality

$$\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)} \le \frac{C_p(\nu, \eta) I^{1/\widehat{q}^*}}{(2\pi)^{d\widehat{\gamma}/2}} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\widehat{\gamma}} \max_{1 \le j \le d} \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\widehat{\gamma}}$$

holds.

As functions $\psi(\cdot)$ defining the operator Λ we can consider the functions

$$\psi_\theta(\xi) = (|\xi_1|^\theta + \dots + |\xi_d|^\theta)^{2/\theta}, \quad \theta > 0.$$

The corresponding operator is denoted by Λ_θ. In particular, $\Lambda_2 = -\Delta$, where Δ is the Laplace operator. We denote by $\Lambda_\theta^{\eta/2}$ the operator Λ which is defined by $\psi(\cdot) = \psi_\theta^{\eta/2}(\cdot)$.

Now we consider the case when $p = 2$.

Theorem 7. *Let $\nu > \eta > 0$, $\nu \ge 1$, and $0 < \theta \le 2\nu$. Then*

$$E_{22}(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d})) = d^{\eta/\theta}\left(\frac{\delta}{(2\pi)^{d/2}}\right)^{1-\eta/\nu}, \tag{55}$$

and all methods

$$\widehat{m}(y)(t) = F^{-1}\left(a(t)\psi_\theta^{\eta/2}(t)y(t)\right), \tag{56}$$

where $a(\cdot)$ are measurable functions satisfying the condition

$$\psi_\theta^{\eta}(\xi)\left(\frac{|1-a(\xi)|^2}{\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}} + \frac{|a(\xi)|^2}{(2\pi)^d\lambda_1}\right) \le 1, \tag{57}$$

in which

$$\lambda_1 = \frac{d^{2\eta/\theta}}{(2\pi)^d}\left(1-\frac{\eta}{\nu}\right)\left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu}, \quad \lambda_2 = \frac{\eta}{\nu}d^{2\eta/\theta-1}\left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu-1},$$

are optimal.

The sharp inequality

$$\|\Lambda_\theta^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \le \frac{d^{\eta/\theta}\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu}}{(2\pi)^{d(1-\eta/\nu)/2}}\max_{1\le j\le d}\|D^{\nu e_j}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\eta/\nu} \tag{58}$$

holds.

Proof. It follows by Lemma 1 that

$$E_{22}(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d})) \ge \sup_{\substack{x(\cdot)\in W_2^{(D^{\nu e_1},\dots,D^{\nu e_d})} \\ \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta}} \|\Lambda_\theta^{\eta/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}. \tag{59}$$

Given $0 < \varepsilon < (2\pi)^{d/(2\nu)}\delta^{-1/\nu}$, we set

$$\widehat{\xi}_\varepsilon = \left(\frac{(2\pi)^d}{\delta^2}\right)^{\frac{1}{2\nu}}(1,\dots,1) - (\varepsilon,\dots,\varepsilon), \quad B_\varepsilon = \{\xi\in\mathbb{R}^d : |\xi - \widehat{\xi}_\varepsilon| < \varepsilon\}.$$

Consider a function $x_\varepsilon(\cdot)$ such that

$$Fx_\varepsilon(\xi) = \begin{cases} \dfrac{\delta}{\sqrt{\operatorname{mes} B_\varepsilon}}, & \xi\in B_\varepsilon, \\ 0, & \xi\notin B_\varepsilon. \end{cases} \tag{60}$$

Then $\|Fx_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \delta^2$ and

$$\|D^{\nu e_j}x_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{\delta^2}{(2\pi)^d\operatorname{mes} B_\varepsilon}\int_{B_\varepsilon}|\xi_j|^{2\nu}\,d\xi \le 1, \ j = 1,\dots,d.$$

By virtue of (59) we have

$$E_{22}^2(\Lambda_\theta^{\eta/2},(D^{\nu e_1},\dots,D^{\nu e_d}))\ge\|\Lambda_\theta^{\eta/2}x_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$=\frac{\delta^2}{(2\pi)^d\,\mathrm{mes}\,B_\varepsilon}\int\limits_{B_\varepsilon}\psi_\theta^\eta(\xi)\,d\xi=\frac{\delta^2}{(2\pi)^d}\psi_\theta^\eta(\widetilde{\xi}_\varepsilon),\quad \widetilde{\xi}_\varepsilon\in B_\varepsilon.$$

Letting $\varepsilon\to0$ we obtain the estimate

$$E_{22}^2(\Lambda_\theta^{\eta/2},(D^{\nu e_1},\dots,D^{\nu e_d}))\ge d^{2\eta/\theta}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-\eta/\nu}. \tag{61}$$

We will find optimal methods among methods (56). Passing to the Fourier transform we have

$$\|\Lambda_\theta^{\eta/2}x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\psi_\theta^\eta(\xi)\,|Fx(\xi)-a(\xi)y(\xi)|^2\,d\xi.$$

We set $z(\cdot)=Fx(\cdot)-y(\cdot)$ and note that

$$\int\limits_{\mathbb{R}^d}|z(\xi)|^2\,d\xi\le\delta^2,\quad \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\xi_j|^{2\nu}|Fx(\xi)|^2\,d\xi\le1,\ j=1,\dots,d.$$

Then

$$\|\Lambda_\theta^{\eta/2}x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\psi_\theta^\eta(\xi)\,|(1-a(\xi))\,Fx(\xi)+a(\xi)z(\xi)|^2\,d\xi.$$

We write the integrand as

$$\psi_\theta^\eta(\xi)\left|\frac{(1-a(\xi))\sqrt{\lambda_2}\left(\sum_{j=1}^d|\xi_j|^{2\nu}\right)^{1/2}Fx(\xi)}{\sqrt{\lambda_2}\left(\sum_{j=1}^d|\xi_j|^{2\nu}\right)^{1/2}}+\frac{a(\xi)}{(2\pi)^{d/2}\sqrt{\lambda_1}}(2\pi)^{d/2}\sqrt{\lambda_1}z(\xi)\right|^2.$$

Applying the Cauchy-Bunyakovskii-Schwarz inequality we obtain the estimate

$$\|\Lambda_\theta^{\eta/2}x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$\le\operatorname*{vraisup}_{\xi\in\mathbb{R}^d}S(\xi)\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\left(\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}|Fx(\xi)|^2+(2\pi)^d\lambda_1|z(\xi)|^2\right)d\xi,$$

where

$$S(\xi)=\psi_\theta^\eta(\xi)\left(\frac{|1-a(\xi)|^2}{\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}}+\frac{|a(\xi)|^2}{(2\pi)^d\lambda_1}\right).$$

If we assume that $S(\xi)\le1$ for almost all ξ, then taking into account (61), we get

$$e_{22}^2(\Lambda_\theta^{\eta/2},(D^{\nu e_1},\dots,D^{\nu e_d}),\widehat{m})$$
$$\le\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\left(\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}|Fx(\xi)|^2+(2\pi)^d\lambda_1|z(\xi)|^2\right)d\xi\le\lambda_2d+\lambda_1\delta^2$$
$$=d^{2\eta/\theta}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-\eta/\nu}\le E_{22}^2(\Lambda_\theta^{\eta/2},(D^{\nu e_1},\dots,D^{\nu e_d})).$$

This proves (55) and shows that the methods under consideration are optimal.

It remains to verify that the set of functions $a(\cdot)$ satisfying (57) is nonempty. Put

$$a(\xi)=\frac{(2\pi)^d\lambda_1}{(2\pi)^d\lambda_1+\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}}.$$

Then

$$S(\xi)=\frac{\psi_\theta^\eta(\xi)}{(2\pi)^d\lambda_1+\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}}.$$

Since $\theta\le 2\nu$ by Hölder's inequality

$$\sum_{j=1}^d|\xi_j|^\theta\le\left(\sum_{j=1}^d|\xi_j|^{2\nu}\right)^{\theta/(2\nu)}d^{1-\theta/(2\nu)}.$$

Putting $\rho=(|\xi_1|^\theta+\ldots+|\xi_d|^\theta)^{1/\theta}$, we obtain

$$\sum_{j=1}^d|\xi_j|^{2\nu}\ge\rho^{2\nu}d^{1-2\nu/\theta}.$$

Thus,

$$S(\xi)\le\frac{\rho^{2\eta}}{(2\pi)^d\lambda_1+\lambda_2\rho^{2\nu}d^{1-2\nu/\theta}}.$$

It is easily checked that the function $f(\rho)=(2\pi)^d\lambda_1+\lambda_2\rho^{2\nu}d^{1-2\nu/\theta}-\rho^{2\eta}$ reaches a minimum on $[0,+\infty)$ at

$$\rho_0=d^{1/\theta}\left(\frac{(2\pi)^d}{\delta^2}\right)^{1/(2\nu)}.$$

Moreover, $f(\rho_0)=0$. Consequently, $f(\rho)\ge0$ for all $\rho\ge0$. Hence $S(\xi)\le1$ for all ξ.

Inequality (58) is proved by the analogy with the proof of Corollary 1. $\square$

We give a simple example of sharp inequality (58) for $\theta=\eta=2$. In this case for any integer $\nu>2$ we have

$$\|\Delta x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\frac{d}{(2\pi)^{d(1-2/\nu)/2}}\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-2/\nu}\max_{1\le j\le d}\left\|\frac{\partial^\nu x}{\partial t_j^\nu}(\cdot)\right\|_{L_2(\mathbb{R}^d)}^{2/\nu}$$

or (since $\|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}=(2\pi)^{d/2}\|x(\cdot)\|_{L_2(\mathbb{R}^d)}$)

$$\|\Delta x(\cdot)\|_{L_2(\mathbb{R}^d)}\le d\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-2/\nu}\max_{1\le j\le d}\left\|\frac{\partial^\nu x}{\partial t_j^\nu}(\cdot)\right\|_{L_2(\mathbb{R}^d)}^{2/\nu}.$$

3.2. Recovery in the metric $L_\infty(\mathbb{R}^d)$

Put

$$\gamma_1=\frac{\nu-\eta-d/2}{\nu+d(1/2-1/p)},\quad q_1=\frac{1}{1/2+\gamma_1(1/2-1/p)},$$

$$\widetilde{C}_p(\nu,\eta)=\gamma_1^{-\frac{\gamma_1}{p}}\left(\frac{1-\gamma_1}{n}\right)^{-\frac{1-\gamma_1}{2}}\left(\frac{B\left(q_1\gamma_1/p+1,q_1(1-\gamma_1)/2\right)}{2|\nu-\eta-d/2|}\right)^{1/q_1}.$$

For $1<p<\infty$ we define $k(\cdot)$ by the equality

$$\frac{k(t)}{(1-k(t))^{p-1}}=(2\pi)^d\frac{|\psi(t)|^{p-2}}{s_2^{p-1}(t)}.$$

We set

$$k(t)=\begin{cases}\min\left\{1,(2\pi)^d|\psi(t)|^{-1}\right\}, & p=1,\\ \left(1-s_2(t)|\psi(t)|^{-1}\right)_+, & p=\infty.\end{cases}$$

Theorem 8. *Let* $1\le p\le\infty$, $\gamma_1\in(0,1)$. *Assume that*

$$I=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{q_1}(\omega)}{\widetilde{s}_2^{q_1(1-\gamma_1)/2}(\omega)}J(\omega)\,d\omega<\infty$$

and $I_1'=\ldots=I_n'$, *where*

$$I_j'=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{q_1}(\omega)\widetilde{\varphi}_j^2(\omega)}{\widetilde{s}_2^{q_1(1-\gamma_1)/2+1}(\omega)}J(\omega)\,d\omega,\ \ j=1,\ldots,n.$$

Then

$$E_{p\infty}(\Lambda,\mathcal{D})=\frac{1}{(2\pi)^{d(1+\gamma_1)/2}}\widetilde{C}_p(\nu,\eta)I^{1/q_1}\delta^{\gamma_1}.$$

The method

$$\widehat{m}(y)(t)=F^{-1}\left(k\left(\xi_1^{\frac{1}{n+d(1/2-1/p)}}t\right)\psi(t)y(t)\right),$$

where

$$\xi_1=\delta\gamma_1^{-\frac{q_1}{2p}}\left(\frac{(1-\gamma_1)\widetilde{C}_p(\nu,\eta)I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}}\right)^{q_1(1/2-1/p)},$$

is optimal.

The sharp inequality

$$\|\Lambda x(\cdot)\|_{L_\infty(\mathbb{R}^d)}\le\frac{\widetilde{C}_p(\nu,\eta)I^{1/q_1}}{(2\pi)^{d(1+\gamma_1)/2}}\|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\gamma_1}\max_{1\le j\le n}\|D_jx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\gamma_1} \tag{62}$$

holds.

Proof. Using an estimate similar to (52) we have

$$E_{p\infty}(\Lambda,\mathcal{D})\ge\sup_{\substack{x(\cdot)\in W_p^{\mathcal{D}}\\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}}\|\Lambda x(\cdot)\|_{L_\infty(\mathbb{R}^d)}.$$

Assume that $x(\cdot)\in W_p^{\mathcal{D}}$ and $\|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta$. If $\widehat{x}(\cdot)$ is such that $F\widehat{x}(\xi)=\varepsilon(\xi)e^{-i\langle t,\xi\rangle}Fx(\xi)$, where

$$\varepsilon(\xi)=\begin{cases}\dfrac{\overline{\psi(\xi)Fx(\xi)}}{|\psi(\xi)Fx(\xi)|}, & \psi(\xi)Fx(\xi)\ne0,\\ 0, & \psi(\xi)Fx(\xi)=0,\end{cases}$$

then we obtain $\widehat{x}(\cdot)\in W_p^{\mathcal{D}}$, $\|F\widehat{x}(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta$ and

$$\left|\int_{\mathbb{R}^d} \psi(\xi)F\widehat{x}(\xi)e^{i\langle t,\xi\rangle}\,d\xi\right| = \int_{\mathbb{R}^d} |\psi(\xi)Fx(\xi)|\,d\xi.$$

Hence

$$E_{p\infty}(\Lambda,\mathcal{D}) \ge \frac{1}{(2\pi)^d} \sup_{\substack{x(\cdot)\in W_p^{\mathcal{D}} \\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta}} \int_{\mathbb{R}^d} |\psi(\xi)Fx(\xi)|\,d\xi. \tag{63}$$

Let $1\le p<\infty$. It follows from (22) that

$$E_{p\infty}(\Lambda,\mathcal{D}) \ge E(p,1,2),$$

where, in the problem of the evaluation of $E(p,1,2)$, the functions $\varphi_j(\cdot)$ should be replaced by the function $(2\pi)^{-d/2}\varphi_j(\cdot)$, and the function $\psi(\cdot)$ by $(2\pi)^{-d}\psi(\cdot)$. From Theorem 5 we obtain

$$E_{p\infty}(\Lambda,\mathcal{D}) \ge \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} K\delta^{\gamma_1},$$

where

$$K = \gamma_1^{-\frac{\gamma_1}{p}} \left(\frac{1-\gamma_1}{n}\right)^{-\frac{1-\gamma_1}{2}} \left(\frac{B\,(q_1\gamma_1/p, q_1(1-\gamma_1)/2)\,I}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)}\right)^{1/q_1}.$$

From the properties of the beta-function we have

$$\begin{aligned}\frac{B\,(q_1\gamma_1/p, q_1(1-\gamma_1)/2)}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)} &= \frac{B\,(q_1\gamma_1/p+1, q_1(1-\gamma_1)/2)\,(q_1\gamma_1/p+q_1(1-\gamma_1)/2)}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)q_1\gamma_1/p} \\ &= \frac{B\,(q_1\gamma_1/p+1, q_1(1-\gamma_1)/2)}{2|\nu-\eta-d/2|}.\end{aligned}$$

Thus,

$$E_{p\infty}(\Lambda,\mathcal{D}) \ge \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_p(\nu,\eta) I^{1/q_1}\delta^{\gamma_1}.$$

Moreover, it follows from the same Theorem 5 that

$$\int_{\mathbb{R}^d} \left|\frac{1}{(2\pi)^d}\psi(\xi)F(\xi) - m(y)(\xi)\right| d\xi \le E(p,1,2),$$

where

$$m(y)(t) = \frac{1}{(2\pi)^d} k\left(\xi_1^{\frac{1}{\nu+d(1/2-1/p)}} t\right)\psi(t)y(t),$$

and

$$\begin{aligned}\xi_1 = \frac{\delta}{\gamma_1^{1/p}} \left(\frac{1-\gamma_1}{n}\right)^{1/2} &\left(\frac{B\,(q_1\gamma_1/p, q_1(1-\gamma_1)/2)\,I(2\pi)^{-dq_1(1+\gamma_1)/2}}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)}\right)^{1/2-1/p} \\ &= \delta\gamma_1^{-\frac{q_1}{2p}} \left(\frac{(1-\gamma_1)\widetilde{C}_p(\nu,\eta)I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}}\right)^{q_1(1/2-1/p)}.\end{aligned}$$

Consequently,

$$\left|\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\psi(\xi)F(\xi)e^{i\langle t,\xi\rangle}\,d\xi-\int\limits_{\mathbb{R}^d}m(y)(\xi)e^{i\langle t,\xi\rangle}\,d\xi\right|$$

$$\le\int\limits_{\mathbb{R}^d}\left|\frac{1}{(2\pi)^d}\psi(\xi)F(\xi)-m(y)(\xi)\right|d\xi\le E(p,1,2)\le E_{p\infty}(\Lambda,\mathcal{D}).$$

It follows that the method $\widehat{m}(y)(\cdot)$ is optimal, and the error of optimal recovery coincides with $E(p,1,2)$.

Now we consider the case when $p=\infty$. Put

$$s(\xi)=\begin{cases}\dfrac{\psi(\xi)}{|\psi(\xi)|}, & \psi(\xi)\ne 0,\\ 1, & \psi(\xi)=0.\end{cases}$$

Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi)=\begin{cases}\delta\overline{s(\xi)}, & |\psi(\xi)|\ge\lambda s_2(\xi),\\ \dfrac{\delta\overline{\psi(\xi)}}{\lambda s_2(\xi)}, & |\psi(\xi)|<\lambda s_2(\xi).\end{cases}$$

We choose $\lambda>0$ such that

$$\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\varphi_j(\xi)|^2|F\widehat{x}(\xi)|^2\,d\xi=1,\quad j=1,\dots,n.$$

Now, to find λ we have the equation

$$\frac{\delta^2}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}|\varphi_j(\xi)|^2\,d\xi+\frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int\limits_{|\psi(\xi)|<\lambda s_2(\xi)}\frac{|\varphi_j(\xi)|^2|\psi(\xi)|^2}{s_2^2(\xi)}\,d\xi=1.$$

If $\nu>\eta+d/2$, then from the fact that $\gamma_1\in(0,1)$ it follows that $\eta>-d$. In this case it is easy to check that $2\nu>\eta$ and $2\nu+d>0$. Passing to the polar transformation we obtain

$$\frac{\delta^2}{(2\pi)^d}\int\limits_{\Pi_{d-1}}\widetilde{\varphi}_j^2(\omega)J(\omega)\,d\omega\int\limits_0^{\Phi_2(\omega)}\rho^{2\nu+d-1}\,d\rho$$

$$+\frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int\limits_{\Pi_{d-1}}\frac{\widetilde{\varphi}_j^2(\omega)\widetilde{\psi}^2(\omega)}{\widetilde{s}_2^2(\omega)}J(\omega)\,d\omega\int\limits_{\Phi_2(\omega)}^{+\infty}\rho^{-2\nu+2\eta+d-1}\,d\rho=1,$$

where

$$\Phi_2(\omega)=\left(\frac{\widetilde{\psi}(\omega)}{\lambda\widetilde{s}_2(\omega)}\right)^{\frac{1}{2\nu-\eta}}.$$

Thus,

$$\frac{\delta^2}{(2\pi)^d}\lambda^{-\frac{2\nu+d}{2\nu-\eta}}\frac{4\nu-2\eta}{(2\nu+d)(2\nu-2\eta-d)}I_j=1.$$

If $\nu<\eta+d/2$, then from the fact that $\gamma_1\in(0,1)$ it follows that $\eta<-d$, $2\nu<\eta$, and $2\nu+d<0$. Passing to the polar transformation we obtain

$$\frac{\delta^2}{(2\pi)^d}\int\limits_{\Pi_{d-1}}\widetilde{\varphi}_j^2(\omega)\,J(\omega)\,d\omega\int\limits_{\Phi_2(\omega)}^{+\infty}\rho^{2\nu+d-1}\,d\rho + \frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int\limits_{\Pi_{d-1}}\frac{\widetilde{\varphi}_j^2(\omega)\widetilde{\psi}^2(\omega)}{\widetilde{s}_2^2(\omega)}J(\omega)\,d\omega\int\limits_0^{\Phi_2(\omega)}\rho^{-2\nu+2\eta+d-1}\,d\rho=1.$$

For this case we have

$$\frac{\delta^2}{(2\pi)^d}\lambda^{-\frac{2\nu+d}{2\nu-\eta}}\frac{2\eta-4\nu}{(2\nu+d)(2\nu-2\eta-d)}I_j=1.$$

Combining both of these cases and taking into account that $I_j=I/n$, $j=1,\dots,n$, we get

$$\lambda=\left(\frac{2\delta^2|2\nu-\eta|I}{(2\pi)^d n(2\nu+d)(2\nu-2\eta-d)}\right)^{\frac{2\nu-\eta}{2\nu+d}}.$$

It follows by (63) that

$$E_{\infty\infty}(\Lambda,\mathcal{D})\ge\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\psi(\xi)F\widehat{x}(\xi)|\,d\xi=\frac{\delta}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}|\psi(\xi)|\,d\xi + \frac{\delta}{\lambda(2\pi)^d}\int\limits_{|\psi(\xi)|<\lambda s_2(\xi)}\frac{|\psi(\xi)|^2}{s_2(\xi)}\,d\xi.$$

Using calculations similar to those that were above, we obtain

$$E_{\infty\infty}(\Lambda,\mathcal{D})\ge\frac{\delta|2\nu-\eta|\lambda^{-\frac{\eta+d}{2\nu-\eta}}I}{(2\pi)^d(\eta+d)(2\nu-2\eta-d)}=E_0,$$

where

$$E_0=\frac{(n|\nu+d/2|)^{\frac{\eta+d}{2\nu+d}}}{\eta+d}\left(\frac{(2\nu-\eta)I}{(2\pi)^d(2\nu-2\eta-d)}\right)^{\frac{2\nu-\eta}{2\nu+d}}\delta^{\frac{2\nu-2\eta-d}{2\nu+d}}.$$

We prove that for all $x(\cdot)\in X_\infty$ the equality

$$\Lambda x(t)=\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi + \frac{\lambda}{\delta(2\pi)^d}\int\limits_{\mathbb{R}^d}s_2(\xi)Fx(\xi)\overline{F\widehat{x}(\xi)}e^{i\langle t,\xi\rangle}\,d\xi \tag{64}$$

holds. Indeed,

$$\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi + \frac{\lambda}{\delta(2\pi)^d}\int\limits_{\mathbb{R}^d}s_2(\xi)Fx(\xi)\overline{F\widehat{x}(\xi)}e^{i\langle t,\xi\rangle}\,d\xi$$

$$=\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}((\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi$$

$$+\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}\lambda s(\xi)s_2(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi$$
$$+\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|<\lambda s_2(\xi)}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi=\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi$$
$$=\Lambda x(t).$$

We estimate the error of the method

$$m(y)(t)=\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,y(\xi)e^{i\langle t,\xi\rangle}\,d\xi.$$

We have

$$|\Lambda x(t)-m(y)(t)|\le\Bigg|\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi$$
$$-\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\Bigg|$$
$$+\frac{1}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}|\psi(\xi)-\lambda s(\xi)s_2(\xi)|\,|Fx(\xi)-y(\xi)|\,d\xi.$$

If $x(\cdot)$ such that

$$\|Fx(\cdot)-y(\cdot)\|_{L_\infty(\mathbb{R}^d)}\le\delta,\quad \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}|\varphi_j(\xi)|^2|Fx(\xi)|^2\,d\xi\le1,\ \ j=1,\dots,n,$$

then, taking into account (64), we obtain

$$|\Lambda x(t)-m(y)(t)|\le\frac{\lambda}{\delta(2\pi)^d}\int\limits_{\mathbb{R}^d}s_2(\xi)|Fx(\xi)||F\widehat{x}(\xi)|\,d\xi+\mu\le\frac{n\lambda}{\delta}+\mu,$$

where

$$\mu=\frac{\delta}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}(|\psi(\xi)|-\lambda s_2(\xi))\,d\xi.$$

Passing to the polar transformation we find

$$\frac{\delta}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}|\psi(\xi)|\,d\xi=\frac{\delta\lambda^{-\frac{\eta+d}{2\nu-\eta}}}{(2\pi)^d|\eta+d|}I,$$
$$\frac{\delta\lambda}{(2\pi)^d}\int\limits_{|\psi(\xi)|\ge\lambda s_2(\xi)}s_2(\xi)\,d\xi=\frac{\delta\lambda^{-\frac{\eta+d}{2\nu-\eta}}}{(2\pi)^d|2\nu+d|}I.$$

Hence

$$\mu=\frac{\delta\lambda^{-\frac{\eta+d}{2\nu-\eta}}|2\nu-\eta|}{(2\pi)^d(\eta+d)(2\nu+d)}I.$$

It is easily checked that $n\lambda/\delta+\mu=E_0$, and therefore

$$e_{\infty\infty}(\Lambda, \mathcal{D}, m) \le E_0 \le E_{\infty\infty}(\Lambda, \mathcal{D}).$$

It follows that $m(y)(\cdot)$ is an optimal method, and the error of optimal recovery is E_0. It is easily checked that for $p = \infty$

$$\frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_\infty(\nu, \eta) I^{1/q_1} \delta^{\gamma_1} = E_0.$$

We evaluate ξ_1 for $p = \infty$. We have

$$\xi_1 = \delta \left(\frac{(1-\gamma_1)\widetilde{C}_\infty(\nu, \eta) I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}} \right)^{q_1/2} = \lambda^{\frac{\nu+d/2}{2\nu-\eta}}. \tag{65}$$

The method $m(y)(\cdot)$ can be written as

$$m(y)(t) = F^{-1}\left(\left(1 - \lambda \frac{s_2(\xi))}{|\psi(t)|}\right)_+ \psi(t) y(t) \right).$$

In view of (65) we have

$$m(y)(t) = F^{-1}\left(k\left(\xi_1^{\frac{1}{n+d/2}} t\right) \psi(t) y(t) \right) = \widehat{m}(y)(t).$$

Inequality (62) is proved by the analogy with the proof of Corollary 1. □

It is not difficult to formulate a corollary from Theorem 8 analogous to Corollary 7 for the same Λ and $\mathcal{D} = (D^{\nu e_1}, \dots, D^{\nu e_d})$.

References

[1] F.I. Andrianov, Multidimensional analogues of Carlson's inequality and its generalizations, Izv. Vysš. Učebn. Zaved., Mat. 1 (56) (1960) 3–7 (in Russian).
[2] S. Barza, V. Burenkov, J. Pečarić, L.-E. Persson, Sharp multidimentional multiplicative inequalities for weighted L_p spaces with homogeneous weights, Math. Inequal. Appl. 1 (1) (1998) 53–67.
[3] F. Carlson, Une inégalité, Ark. Mat. Astron. Fys. 25B (1934) 1–5.
[4] K. Jichang, L. Debnath, An extension of Carlson's inequality, J. Math. Anal. Appl. 267 (1) (2002) 395–399.
[5] V.I. Levin, Sharp constants in inequalities of Carlson type, Dokl. Akad. Nauk SSSR 59 (1948) 635–638 (in Russian).
[6] M.-J. Luo, R.K. Raina, A new extension of Carlson's inequality, Math. Inequal. Appl. 19 (2) (2016) 417–424.
[7] C.A. Micchelli, T.J. Rivlin, A survey of optimal recovery, in: C.A. Micchelli, T.J. Rivlin (Eds.), Optimal Estimation in Approximation Theory, Plenum Press, New York, 1977, pp. 1–54.
[8] K.Yu. Osipenko, Optimal recovery of linear operators in non-Euclidean metrics, Sb. Math. 205 (10) (2014) 1442–1472.
[9] K.Yu. Osipenko, Optimal recovery of operators and multidimensional Carlson type inequalities, J. Complex. 32 (1) (2016) 53–73.
[10] K.Yu. Osipenko, Inequalities for derivatives with the Fourier transform, Appl. Comput. Harmon. Anal. 53 (2021) 132–150.
[11] K.Yu. Osipenko, Optimal recovery in weighted spaces with homogeneous weights, Sb. Math. 213 (2022) 385–411.
[12] L. Plaskota, Noisy Information and Computational Complexity, Cambridge University Press, Cambridge, 1996.
[13] J.F. Traub, H. Woźniakowski, A General Theory of Optimal Algorithms, Academic Press, New York, 1980.