OPTIMAL RECOVERY AND GENERALIZED CARLSON INEQUALITY FOR WEIGHTS WITH SYMMETRY PROPERTIES

K. YU. OSIPENKO

Abstract. The paper concerns problems of the recovery of operators from noisy information in weighted L_q-spaces with homogeneous weights. A number of general theorems are proved and applied to finding exact constants in multidimensional Carlson type inequalities with several weights and problems of the recovery of differential operators from a noisy Fourier transform. In particular, optimal methods are obtained for the recovery of powers of generalized Laplace operators from a noisy Fourier transform in the L_p-metric.

1. Introduction

Let T be a nonempty set, Σ be the σ-algebra of subsets of T, and μ be a nonnegative σ-additive measure on Σ. We denote by $L_p(T,\Sigma,\mu)$ (or simply $L_p(T,\mu)$) the set of all Σ-measurable functions with values in $\mathbb{R}$ or in $\mathbb{C}$ for which

$$\|x(\cdot)\|_{L_p(T,\mu)}=\left(\int_T |x(t)|^p\,d\mu(t)\right)^{1/p}<\infty,\quad 1\le p<\infty,$$
$$\|x(\cdot)\|_{L_\infty(T,\mu)}=\operatorname*{vraisup}_{t\in T}|x(t)|<\infty,\quad p=\infty.$$

If $T\subset\mathbb{R}^d$ and $d\mu=dt$, $t\in\mathbb{R}^d$, we put $L_p(T)=L_p(T,\mu)$.

The Carlson inequality [3]

$$\|x(t)\|_{L_1(\mathbb{R}_+)}\le\sqrt{\pi}\|x(t)\|^{1/2}_{L_2(\mathbb{R}_+)}\|tx(t)\|^{1/2}_{L_2(\mathbb{R}_+)},\quad \mathbb{R}_+=[0,+\infty),$$

was generalized by many authors (see [4], [1], [2], [8], [9]). In [8] we found sharp constants for inequalities of the form

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le K\|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T,\mu))}\|w_1(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)},$$

where T is a cone in a linear space, $w(\cdot)$, $w_0(\cdot)$, and $w_1(\cdot)$ are homogenous functions and $1\le q<p,r<\infty$ (for $T=\mathbb{R}^d$ the sharp inequality was obtained in [2]). This problem is closely related with the following extremal problem

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)}\to\max,\quad \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu))}\le\delta,\quad \|w_1(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,$$

where $\delta>0$. In this paper we study the extremal problem

$$(1)\quad \|w(\cdot)x(\cdot)\|_{L_q(T,\mu)}\to\max,\quad \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu))}\le\delta,$$
$$\|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n,$$

2010 *Mathematics Subject Classification.* 41A65, 41A46, 49N30.
Key words and phrases. Optimal recovery, Carlson type inequalities, Fourier transform, sharp constants.

where $w(\cdot)$, $w_0(\cdot)$, and $w_j(\cdot)$, $j=1,\dots,n$, are homogenous functions with some symmetry properties. Using the solution of this problem we obtain the sharp constant K for the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le K\|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T,\mu))}\left(\max_{1\le j\le n}\|\omega_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\right)^{1-\gamma}.$$

In particular, we find the sharp constant for the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d_+)}\le C\|w_0(\cdot)x(\cdot)\|^{p\alpha}_{L_p(\mathbb{R}^d_+)}\left(\max_{1\le j\le d}\|\omega_j(\cdot)x(\cdot)\|_{L_r(\mathbb{R}^d_+)}\right)^{r\beta},$$

where $w(t)=(t_1^2+\ldots+t_d^2)^{\theta/2}$, $w_0(t)=(t_1^2+\ldots+t_d^2)^{\theta_0/2}$, $w_j(t)=t_j^{\theta_1}$, $j=1,\dots,d$, $\theta=d(1-1/q)$, $\theta_0=d-(\lambda+d)/p$, $\theta_1=d+(\mu-d)/r$,

$$\alpha=\frac{\mu}{p\mu+r\lambda},\quad \beta=\frac{\lambda}{p\mu+r\lambda},\quad \lambda,\mu>0,$$

and $(p,q,r)\in P\cup P_1\cup P_2$, where

$$P=\{\,(p,q,r):1\le q<p,r\,\},\quad P_1=\{\,(p,q,r):1\le q=r<p\,\},$$
$$P_2=\{\,(p,q,r):1\le q=p<r\,\}.$$

For $d=1$, $q=1$, and $(p,1,r)\in P$ this result was proved in [4] (see also [2]).

It is appeared that the value of (1) is the error of optimal recovery of the operator $\Lambda x(\cdot)=w(\cdot)x(\cdot)$ on the class of functions $x(\cdot)$ such that $\|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1$, $j=1,\dots,n$, by the information about the function $w_0(\cdot)x(\cdot)$ given with the error δ in L_p-norm. Therefore, in section 2 we begin with the setting of optimal recovery problem and then in section 3 we prove some general theorems. In section 4 we consider the case when weights are homogeneous in a cone of linear space and section 5 is devoted to the case of $\mathbb{R}^d$. In section 6 the results obtained are applied to optimal recovery and sharp inequalities of differential operators defined by Fourier transforms.

2. General setting

Let T_0 is not empty μ-measurable subset of T. Put

$$\mathcal{W}=\{\,x(\cdot):x(\cdot)\in L_p(T_0,\mu),\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}<\infty,\ j=1,\dots,n\,\},$$
$$W=\{\,x(\cdot)\in\mathcal{W}:\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n\,\},$$

where $1\le p,r\le\infty$, and $\varphi_j(\cdot)$ is a measurable function on T. Consider the problem of recovery of operator $\Lambda\colon\mathcal{W}\to L_q(T,\mu)$, $1\le q\le\infty$, defined by equality $\Lambda x(\cdot)=\psi(\cdot)x(\cdot)$, where $\psi(\cdot)$ is a measurable function on T, on the class W by the information about functions $x(\cdot)\in W$ given inaccurately (we assume that $\psi(\cdot)$ and $\varphi_j(\cdot)$, $j=1,\dots,n$, such that Λ maps $\mathcal{W}$ to $L_q(T,\mu)$). More precisely, we assume that for any function $x(\cdot)\in W$ we know $y(\cdot)\in L_p(T_0,\mu)$ such that $\|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta$, $\delta>0$. We want to approximate the value $\Lambda x(\cdot)$ knowing $y(\cdot)$. As recovery methods we consider all possible mappings $m\colon L_p(T_0,\mu)\to L_q(T,\mu)$. The error of a method m is defined as

$$e(p,q,r,m)=\sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu)\\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}}\|\Lambda x(\cdot)-m(y)(\cdot)\|_{L_q(T,\mu)}.$$

The quantity

$$E(p,q,r)=\inf_{m\colon L_p(T_0,\mu)\to L_q(T,\mu)}e(p,q,r,m) \tag{2}$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal. Various settings of optimal recovery theory and examples of such problems may be found in [5], [13], [12], [6], [11].

For the lower bound of $E(p,q,r)$ we use the following result which was proved (in more or less general forms) in many papers (see, for example, [7]).

Lemma 1.

$$E(p,q,r) \ge \sup_{\substack{x(\cdot)\in W \\ \|x(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \|\Lambda x(\cdot)\|_{L_q(T,\mu)}. \tag{3}$$

3. Main results

Set

$$\chi_0(t) = \begin{cases} 1, & t\in T_0, \\ 0, & t\notin T_0, \end{cases} \qquad \sigma_r(t) = \sum_{j=1}^n \lambda_j |\varphi_j(t)|^r.$$

Theorem 1. *Let $1 \le q < p, r$, $\lambda_j \ge 0$, $j = 0,1,\dots,n$, $\lambda_0 + \sigma_r(t) \ne 0$ for almost all $t\in T_0$, $\sigma_r(t)\ne 0$ for almost all $t \in T\setminus T_0$, $\widehat{x}(t)\ge 0$ be a solution of equation*

$$-q|\psi(t)|^q + p\lambda_0 x^{p-q}(t)\chi_0(t) + r\sigma_r(t)x^{r-q}(t) = 0, \tag{4}$$

$\overline{\lambda}$ such that

$$\int_{T_0} \widehat{x}^p(t)\,d\mu(t) \le \delta^p, \qquad \int_T |\varphi_j(t)|^r \widehat{x}^r(t)\,d\mu(t) \le 1,\ j=1,\dots,n, \tag{5}$$

$$\lambda_0\left(\int_{T_0} \widehat{x}^p(t)\,d\mu(t) - \delta^p\right) = 0, \quad \lambda_j\left(\int_T |\varphi_j(t)|^r \widehat{x}^r(t)\,d\mu(t) - 1\right) = 0,\ j=1,\dots,n.$$

Then

$$E(p,q,r) = \left(q^{-1}p\lambda_0\delta^p + q^{-1}r\sum_{j=1}^n \lambda_j\right)^{1/q}, \tag{6}$$

and the method

$$\widehat{m}(y)(t) = \begin{cases} q^{-1}p\lambda_0\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t), & t\in T_0,\ \psi(t)\ne 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{7}$$

is optimal recovery method.

To prove this theorem we need some preliminary results. The first one is actually a sufficient condition in the Kuhn-Tucker theorem (the only difference is that we do not require convexity of functions).

Let $f_j\colon A\to\mathbb{R}$, $j=0,1,\dots,k$, be functions defined on some set A. Consider the extremal problem

$$f_0(x)\to\max, \quad f_j(x)\le 0, \quad j=1,\dots,k, \quad x\in A, \tag{8}$$

and write down its Lagrange function

$$\mathcal{L}(x,\lambda) = -f_0(x) + \sum_{j=1}^k \lambda_j f_j(x), \quad \lambda = (\lambda_1,\dots,\lambda_k).$$

Lemma 2. *Assume that there exist* $\widehat{\lambda}_j \ge 0$, $j = 1, \dots, k$, *and an element* $\widehat{x} \in A$, *admissible for problem* (8), *such that*

$$(a) \quad \min_{x \in A} \mathcal{L}(x, \widehat{\lambda}) = \mathcal{L}(\widehat{x}, \widehat{\lambda}), \quad \widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_k),$$

$$(b) \quad \widehat{\lambda}_j f_j(\widehat{x}) = 0, \ j = 1, \dots, k.$$

Then $\widehat{x}$ *is an extremal element for problem* (8).

Proof. For any x admissible for problem (8) we have

$$-f_0(x) \ge \mathcal{L}(x, \widehat{\lambda}) \ge \mathcal{L}(\widehat{x}, \widehat{\lambda}) = -f_0(\widehat{x}).$$

□

Put

$$F(u, v, \alpha) = -((1-\alpha)u + \alpha v)^q + av^p + bu^r, \quad u, v \ge 0, \quad \alpha \in [0, 1],$$

where $a, b \ge 0$, and $1 \le p, q, r < \infty$.

Lemma 3 ([8])**.** *For all* $a, b \ge 0$, $a + b > 0$, *and all* $1 \le q < p, r < \infty$, *there exists the unique solution* $\widehat{u} > 0$ *of the equation*

$$-q + pau^{p-q} + rbu^{r-q} = 0.$$

Moreover, for all $u, v \ge 0$ *and* $\alpha = q^{-1}pa\widehat{u}^{p-q} = 1 - q^{-1}rb\widehat{u}^{r-q}$

$$F(\widehat{u}, \widehat{u}, \alpha) \le F(u, v, \alpha).$$

In particular, for all $u \ge 0$

$$-\widehat{u}^q + a\widehat{u}^p + b\widehat{u}^r \le -u^q + au^p + bu^r.$$

Proof of Theorem 1. 1. Lower estimate. The extremal problem on the right-hand side of (3) (for convenience, we raise the quantity to be maximized to the q-th power) is as follows:

$$(9) \quad \int_T |\psi(t)x(t)|^q \, d\mu(t) \to \max, \quad \int_{T_0} |x(t)|^p \, d\mu(t) \le \delta^p,$$
$$\int_T |\varphi_j(t)x(t)|^r \, d\mu(t) \le 1, \ j = 1, \dots, n.$$

If $t \in T$ such that $\psi(t) = 0$, then evidently $\widehat{x}(t) = 0$. If $\psi(t) \ne 0$ we obtain by Lemma 3 that that there is the unique solution $\widehat{x}(t)$ of (4). It follows by (5) that $\widehat{x}(\cdot)$ is admissible function for problem (9). Therefore, by (3) we obtain

$$E(p, q, r) \ge \left(\int_T |\psi(t)|^q \widehat{x}^q(t) \, d\mu(t) \right)^{1/q}.$$

From (4) we have

$$|\psi(t)|^q \widehat{x}^q(t) = q^{-1}p\lambda_0 \widehat{x}^p(t)\chi_0(t) + q^{-1}r\sigma_r(t)\widehat{x}^r(t).$$

Integrating this equality over the set T, we obtain

$$\int_T |\psi(t)|^q \widehat{x}^q(t) \, d\mu(t) = q^{-1}p\lambda_0\delta^p + q^{-1}r\sum_{j=1}^n \lambda_j.$$

Thus,

$$E(p, q, r) \ge \left(q^{-1}p\lambda_0\delta^p + q^{-1}r\sum_{j=1}^n \lambda_j \right)^{1/q}.$$

2. Upper estimate. To estimate the error of method (7) we need to find the value of the extremal problem:

$$(10)\quad \int_{T_0} |\psi(t)x(t)-\psi(t)\alpha(t)y(t)|^q\,d\mu(t)+\int_{T\setminus T_0}|\psi(t)x(t)|^q\,d\mu(t)\to\max,$$
$$\int_{T_0}|x(t)-y(t)|^p\,d\mu(t)\le\delta^p,\quad \int_T|\varphi_j(t)x(t)|^r\,d\mu(t)\le 1,\ j=1,\dots,n,$$

where

$$\alpha(t)=\begin{cases}q^{-1}p\lambda_0\widehat{x}^{p-q}(t)|\psi(t)|^{-q}, & t\in T_0,\ \psi(t)\ne 0,\\ 0, & \text{otherwise.}\end{cases}$$

Put

$$z(t)=\begin{cases}x(t)-y(t), & t\in T_0,\\ 0, & t\in T\setminus T_0.\end{cases}$$

Then (10) may be rewritten as follows:

$$\int_T |\psi(t)|^q|(1-\alpha(t))x(t)+\alpha(t)z(t)|^q\,d\mu(t)\to\max,$$
$$\int_{T_0}|z(t)|^p\,d\mu(t)\le\delta^p,\quad \int_T|\varphi_j(t)x(t)|^r\,d\mu(t)\le 1,\ j=1,\dots,n.$$

The value of this problem does not exceed the value of the problem

$$(11)\quad \int_T|\psi(t)|^q((1-\alpha(t))u(t)+\alpha(t)v(t))^q\,d\mu(t)\to\max,$$
$$\int_{T_0}v^p(t)\,d\mu(t)\le\delta^p,\quad \int_T|\varphi_j(t)|^r u^r(t)\,d\mu(t)\le 1,\ j=1,\dots,n,$$
$$u(t)\ge 0,\ v(t)\ge 0\quad\text{for almost all}\quad t\in T.$$

The Lagrange function for this problem is

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})=\int_T L(t,u(t),v(t),\overline{\lambda})\,d\mu(t),$$

where

$$L(t,u,v,\overline{\lambda})=-|\psi(t)|^q((1-\alpha(t))u+\alpha(t)v)^q+\lambda_0v^p\chi_0(t)+\sigma_r(t)u^r.$$

By Lemma 3 we have

$$L(t,\widehat{x}(t),\widehat{x}(t),\overline{\lambda})\le L(t,u(t),v(t),\overline{\lambda}).$$

Thus,

$$\mathcal{L}(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda})\le\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda}).$$

It follows by Lemma 2 that functions $u(\cdot)=v(\cdot)=\widehat{x}(\cdot)$ are extremal in (11). Consequently,

$$e(p,q,r,\widehat{m})\le\left(\int_T|\psi(t)|^q\widehat{x}^q(t)\,d\mu(t)\right)^{1/q}=\left(q^{-1}p\lambda_0\delta^p+q^{-1}r\sum_{j=1}^n\lambda_j\right)^{1/q}\le E(p,q,r).$$

It means that method (7) is optimal and equality (6) holds. □

Denote $a_+=\max\{a,0\}$.

Theorem 2. *Let* $1 \le q = r < p$, $\lambda_0 > 0$, $\lambda_j \ge 0$, $j = 1, \dots, n$,

$$\widehat{x}(t) = \begin{cases} \left(\dfrac{q}{p\lambda_0}\left(|\psi(t)|^q - \sigma_q(t)\right)_+\right)^{\frac{1}{p-q}}, & t \in T_0, \\ 0, & t \notin T_0, \end{cases} \tag{12}$$

$\overline{\lambda}$ *satisfies conditions* (5), *and* $|\psi(t)|^q - \sigma_q(t) \le 0$ *for almost all* $t \notin T_0$. *Then*

$$E(p,q,q) = \left(q^{-1}p\lambda_0\delta^p + \sum_{j=1}^n \lambda_j\right)^{1/q}, \tag{13}$$

and the method

$$\widehat{m}(y)(t) = \begin{cases} \left(1 - |\psi(t)|^{-q}\sigma_q(t)\right)_+ \psi(t)y(t), & t \in T_0,\ \psi(t) \ne 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{14}$$

is optimal.

Proof. 1. Lower estimate. It follows by (5) that $\widehat{x}(\cdot)$ is admissible function for extremal problem in the right-hand side of (3). Therefore,

$$E(p,q,q) \ge \left(\int_T |\psi(t)|^q \widehat{x}^q(t)\, d\mu(t)\right)^{1/q}.$$

From the definition of $\widehat{x}(\cdot)$ we have

$$|\psi(t)|^q \widehat{x}^q(t) = q^{-1}p\lambda_0 \widehat{x}^p(t)\chi_0(t) + \sigma_q(t)\widehat{x}^q(t).$$

Integrating this equality, we obtain

$$\int_T |\psi(t)|^q \widehat{x}^q(t)\, d\mu(t) = q^{-1}p\lambda_0\delta^p + \sum_{j=1}^n \lambda_j.$$

Thus,

$$E(p,q,q) \ge \left(q^{-1}p\lambda_0\delta^p + \sum_{j=1}^n \lambda_j\right)^{1/q}.$$

2. Upper estimate. Put

$$\alpha(t) = \begin{cases} \left(1 - |\psi(t)|^{-q}\sigma_q(t)\right)_+, & t \in T_0,\ \psi(t) \ne 0, \\ 0, & \text{otherwise}, \end{cases}$$

To estimate the error of method (14) we need to find the value of the extremal problem:

$$\int_{T_0} |\psi(t)|^q |x(t) - \alpha(t)y(t)|^q\, d\mu(t) + \int_{T\setminus T_0} |\psi(t)x(t)|^q\, d\mu(t) \to \max,$$
$$\int_{T_0} |x(t) - y(t)|^p\, d\mu(t) \le \delta^p, \quad \int_T |\varphi_j(t)x(t)|^q\, d\mu(t) \le 1,\ j = 1, \dots, n.$$

Putting $z(\cdot) = x(\cdot) - y(\cdot)$ this problem may be rewritten in the following form

$$\int_{T_0} |\psi(t)|^q |(1-\alpha(t))x(t) + \alpha(t)z(t)|^q\, d\mu(t) + \int_{T\setminus T_0} |\psi(t)x(t)|^q\, d\mu(t) \to \max,$$
$$\int_{T_0} |z(t)|^p\, d\mu(t) \le \delta^p, \quad \int_T |\varphi_j(t)x(t)|^q\, d\mu(t) \le 1,\ j = 1, \dots, n.$$

The value of this problem evidently coincides with the value of the problem

$$\int_T |\psi(t)|^q((1-\alpha(t))v(t)+\alpha(t)u(t))^q\,d\mu(t)\to\max, \tag{15}$$

$$\int_{T_0} u^p(t)\,d\mu(t)\le\delta^p,\quad \int_T |\varphi_j(t)|^q v^q(t)\,d\mu(t)\le 1,\ j=1,\dots,n,$$

$$u(t),v(t)\ge 0,\ \text{for almost all } t\in T.$$

The Lagrange function of (15) has the form

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})=\int_T L(u(t),v(t),\overline{\lambda})\,d\mu(t),$$

where

$$L(u,v,\overline{\lambda})=\begin{cases}-|\psi(t)|^q((1-\alpha(t))v+\alpha(t)u)^q+\lambda_0u^p+\sigma_q(t)v^q, & t\in T_0,\\ -|\psi(t)|^qv^q+\sigma_q(t)v^q, & t\notin T_0.\end{cases}$$

If $\alpha(t)>0$, then

$$\frac{\partial L}{\partial v}=q(v^{q-1}-((1-\alpha(t))v+\alpha(t)u)^{q-1})\sigma_q(t).$$

Therefore, for $\alpha(t)>0$ and any $u>0$, the function $L(u,v,\overline{\lambda})$, $v\in(0,+\infty)$, reaches a minimum at $v=u$. Set $T_0'=\{\,t\in T_0:\alpha(t)>0\,\}$. We have

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})\ge\int_{T_0'}L(u(\cdot),u(\cdot),\overline{\lambda})\,d\mu(t).$$

It is easily checked that for $t\in T_0'$ for all $u(t)\ge 0$

$$L(u(\cdot),u(\cdot),\overline{\lambda})\ge L(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda}).$$

Consequently,

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})\ge\int_{T_0'}L(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda})\,d\mu(t)=\mathcal{L}(\widehat{x}(\cdot),\widehat{x}(\cdot),\overline{\lambda}).$$

Taking into account (5) we obtain by Lemma 2 that $u(\cdot)=v(\cdot)=\widehat{x}(\cdot)$ are extremal functions in (15). Thus,

$$e^q(p,q,q,\widehat{m})=\int_T|\psi(t)\widehat{x}(t)|^q\,d\mu(t)=q^{-1}p\lambda_0\delta^p+\sum_{j=1}^n\lambda_j\le E^q(p,q,q).$$

It means that the method $\widehat{m}$ is optimal and the optimal recovery error is as stated. □

Theorem 3. *Let $1\le q=p<r$, $\lambda_0>0$, $\lambda_j\ge 0$, $j=1,\dots,n$, $\sigma_r(t)\ne 0$ for almost all $t\in T$,*

$$\widehat{x}(t)=\begin{cases}\left(pr^{-1}\sigma_r^{-1}(t)(|\psi(t)|^p-\lambda_0)_+\right)^{\frac{1}{r-p}}, & t\in T_0,\\ \left(pr^{-1}\sigma_r^{-1}(t)|\psi(t)|^p\right)^{\frac{1}{r-p}}, & t\in T\setminus T_0,\end{cases} \tag{16}$$

and $\overline{\lambda}$ satisfies conditions (5). *Then*

$$E(p,p,r)=\left(\lambda_0\delta^p+\frac{r}{p}\sum_{j=1}^n\lambda_j\right)^{1/p}, \tag{17}$$

and the method

$$\widehat{m}(y)(t)=\begin{cases}\alpha(t)\psi(t)y(t), & t\in T_0,\\ 0, & t\in T\setminus T_0,\end{cases}\tag{18}$$

where

$$\alpha(t)=\begin{cases}\min\left\{1,\lambda_0|\psi(t)|^{-p}\right\}, & t\in T_0,\ \psi(t)\neq 0,\\ 0, & \textit{othervise},\end{cases}$$

is optimal.

Proof. 1. Lower estimate. By the definition of $\widehat{x}(\cdot)$ we have

$$|\psi(t)|^p\widehat{x}^p(t)=\lambda_0\widehat{x}^p(t)\chi_0(t)+\frac{r}{p}\sigma_r(t)\widehat{x}^r(t).$$

Using the similar arguments as in the proof of Theorem 1 we obtain

$$E(p,p,r)\ge\left(\int_T|\psi(t)|^p\widehat{x}^p(t)\,d\mu(t)\right)^{1/p}=\left(\lambda_0\delta^p+\frac{r}{p}\sum_{j=1}^n\lambda_j\right)^{1/p}.$$

2. Upper estimate. To estimate the error of method (18) we need to find the value of the following extremal problem:

$$\int_{T_0}|\psi(t)|^p|x(t)-\alpha(t)y(t)|^p\,d\mu(t)+\int_{T\setminus T_0}|\psi(t)x(t)|^p\,d\mu(t)\to\max,$$
$$\int_{T_0}|x(t)-y(t)|^p\,d\mu(t)\le\delta^p,\quad \int_T|\varphi_j(t)x(t)|^r\,d\mu(t)\le 1,\ j=1,\dots,n.$$

Putting $z(\cdot)=x(\cdot)-y(\cdot)$ this problem may be rewritten in the form

$$\int_{T_0}|\psi(t)|^p|(1-\alpha(t))x(t)+\alpha(t)z(t)|^p\,d\mu(t)+\int_{T\setminus T_0}|\psi(t)x(t)|^p\,d\mu(t)\to\max,$$
$$\int_{T_0}|z(t)|^p\,d\mu(t)\le\delta^p,\quad \int_T|\varphi_j(t)x(t)|^r\,d\mu(t)\le 1,\ j=1,\dots,n.$$

The value of this problem evidently coincides with the value of the problem

$$\begin{gathered}\int_T|\psi(t)|^p((1-\alpha(t))v(t)+\alpha(t)u(t))^p\,d\mu(t)\to\max,\\ \int_{T_0}u^p(t)\,d\mu(t)\le\delta^p,\quad \int_T|\varphi_j(t)|^rv^r(t)\,d\mu(t)\le 1,\ j=1,\dots,n,\\ u(t),v(t)\ge 0,\ \text{for almost all } t\in T.\end{gathered}\tag{19}$$

The Lagrange function of (19) has the form

$$\mathcal{L}(u(\cdot),v(\cdot),\overline{\lambda})=\int_T L(u(t),v(t),\overline{\lambda})\,d\mu(t),$$

where

$$L(u,v,\overline{\lambda})=\begin{cases}-|\psi(t)|^p((1-\alpha(t))v+\alpha(t)u)^p+\lambda_0u^p+\sigma_r(t)v^r, & t\in T_0,\\ -|\psi(t)|^pv^p+\sigma_r(t)v^r, & t\in T\setminus T_0.\end{cases}$$

For $t\in T_0$ and $|\psi(t)|^p>\lambda_0$ we have

$$\frac{\partial L}{\partial u}=p\lambda_0(u^{p-1}-((1-\alpha(t))v+\alpha(t)u)^{p-1}).$$

Consequently, in this case for any $v > 0$ the function $L(u, v, \overline{\lambda})$, $v \in (0, +\infty)$, reaches a minimum at $v = u$. If $t \in T_0$, $0 < |\psi(t)|^p \le \lambda_0$, then $\alpha(t) = 1$ and $L(u, v, \overline{\lambda}) \ge 0$. If $t \in T_0$ and $\psi(t) = 0$, then again $L(u, v, \overline{\lambda}) \ge 0$. Set $T_1 = \{t \in T_0 : |\psi(t)|^p > \lambda_0\}$. Then for all $u(t), v(t) \ge 0$ we have

$$\mathcal{L}(u(\cdot), v(\cdot), \overline{\lambda}) \ge \int_{T_1} L(v(\cdot), v(\cdot), \overline{\lambda})\, d\mu(t) + \int_{T\setminus T_0} L(v(\cdot), v(\cdot), \overline{\lambda})\, d\mu(t).$$

It is easy to check that for all $v(t) \ge 0$

$$L(v(\cdot), v(\cdot), \overline{\lambda}) \ge L(\widehat{x}(\cdot), \widehat{x}(\cdot), \overline{\lambda}).$$

Therefore,

$$\mathcal{L}(u(\cdot), v(\cdot), \overline{\lambda}) \ge \int_{T_1\cup(T\setminus T_0)} L(\widehat{x}(\cdot), \widehat{x}(\cdot), \overline{\lambda})\, d\mu(t) = \mathcal{L}(\widehat{x}(\cdot), \widehat{x}(\cdot), \overline{\lambda}).$$

Taking into account (5) we obtain by Lemma 2 that $u(\cdot) = v(\cdot) = \widehat{x}(\cdot)$ are extremal functions in (19). Consequently,

$$e^p(p, p, r, \widehat{m}) = \int_T |\psi(t)\widehat{x}(t)|^q\, d\mu(t) = \lambda_0\delta^p + \frac{r}{p}\sum_{j=1}^n \lambda_j \le E^p(p, p, r).$$

It means that the method $\widehat{m}$ is optimal and the optimal recovery error is as stated. □

Note that if conditions of Theorems 1, 2, and 3 are fulfilled, then we have

$$E(p, q, r) = \sup_{\substack{\|x(\cdot)\|_{L_p(T_0,\mu)}\le\delta \\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n}} \|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}. \tag{20}$$

4. The case of homogenous weight functions

Let T be a cone in a linear space, $T_0 = T$, $\mu(\cdot)$ be a homogenous measure of degree d, $|\psi(\cdot)|$ be homogenous function of degree η, $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, be homogenous functions of degrees ν, $\psi(t) \ne 0$ and $\sum_{j=1}^n |\varphi_j(t)| \ne 0$ for almost all $t \in T$. Let assume, again, that $1 \le p < q, r < \infty$. For $k \in [0, 1)$ the function $k^{\frac{1}{p-q}}(1-k)^{-\frac{1}{r-q}}$ increases monotonically from 0 to $+\infty$. Consequently, there exists $k(\cdot)$ such that for almost all $t \in T$

$$\frac{k^{\frac{1}{p-q}}(t)}{(1-k(t))^{\frac{1}{r-q}}} = s_r^{-\frac{1}{r-q}}(t)|\psi(t)|^{\frac{q(p-r)}{(p-q)(r-q)}}, \quad s_r(t) = \sum_{j=1}^n |\varphi_j(t)|^r. \tag{21}$$

Set

$$k(t) = \begin{cases} \left(1 - |\psi(t)|^{-q}s_q(t)\right)_+, & (p, q, r) \in P_1, \\ \min\{1, |\psi(t)|^{-p}\}. & (p, q, r) \in P_2 \end{cases}$$

Theorem 4. *Let $(p, q, r) \in P \cup P_1 \cup P_2$ and $\nu + d(1/r - 1/p) \ne 0$. Assume that for $(p, q, r) \in P \cup P_1$*

$$I_1 = \int_T |\psi(t)|^{\frac{qp}{p-q}} k^{\frac{p}{p-q}}(t)\, d\mu(t) < \infty,$$
$$I_{j+1} = \int_T |\psi(t)|^{\frac{qr}{p-q}} |\varphi_j(t)|^r k^{\frac{r}{p-q}}(z)\, d\mu(t) < \infty,\ j = 1, \dots, n,$$

and for $(p,q,r)\in P_2$

$$I_1=\int_T\left(s_r^{-1}(t)(|\psi(t)|^p-1)_+\right)^{\frac{p}{r-p}}\,d\mu(t)<\infty,$$

$$I_{j+1}=\int_T|\varphi_j(t)|^r\left(s_r^{-1}(t)(|\psi(t)|^p-1)_+\right)^{\frac{r}{r-p}}\,d\mu(t)<\infty,\ j=1,\dots,n.$$

Moreover, assume that $I_2=\ldots=I_{n+1}$. *Then*

$$E(p,q,r)=\delta^\gamma I_1^{-\gamma/p}I_2^{-(1-\gamma)/r}(I_1+nI_2)^{1/q}, \tag{22}$$

where

$$\gamma=\frac{\nu-\eta-d(1/q-1/r)}{\nu+d(1/r-1/p)}. \tag{23}$$

The method

$$\widehat{m}(y)(t)=k(\xi t)\psi(t)y(t),$$

where

$$\xi=\left(\delta I_1^{-1/p}I_2^{1/r}\right)^{\frac{1}{\nu+d(1/r-1/p)}}, \tag{24}$$

is optimal.

Proof. 1. Let $(p,q,r)\in P$. Put

$$\widehat{x}(t)=\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{1}{p-q}}k^{\frac{1}{p-q}}(\xi t),$$

where λ_0 will be specified later. We have

$$p\lambda_0\widehat{x}^{p-q}(t)=q|\psi(t)|^qk(\xi t) \tag{25}$$

and

$$rc_r(t)\widehat{x}^{r-q}(t)=rc_r(t)\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{r-q}{p-q}}k^{\frac{r-q}{p-q}}(\xi t).$$

Since $|\psi(\cdot)|$ and $|\varphi_j(\cdot)|$, $j=1,\dots,n$, are homogenous it follows by (21) that

$$k^{\frac{r-q}{p-q}}(\xi t)=\frac{|\psi(\xi t)|^{\frac{q(p-r)}{p-q}}}{c_r(\xi t)}(1-k(\xi t))=\xi^{\eta\frac{q(p-r)}{p-q}-\nu r}\frac{|\psi(t)|^{\frac{q(p-r)}{p-q}}}{c_r(t)}(1-k(\xi t)).$$

Thus,

$$rc_r(t)\widehat{x}^{r-q}(t)=r\left(\frac{q}{p\lambda_0}\right)^{\frac{r-q}{p-q}}\xi^{\eta\frac{q(p-r)}{p-q}-\nu r}|\psi(t)|^q(1-k(\xi t)).$$

Put

$$\lambda=\frac{q}{r}\left(\frac{q}{p\lambda_0}\right)^{-\frac{r-q}{p-q}}\xi^{-\eta\frac{q(p-r)}{p-q}+\nu r}. \tag{26}$$

Then

$$r\lambda c_r(t)\widehat{x}^{r-q}(t)=q|\psi(t)|^q(1-k(\xi t)). \tag{27}$$

Taking the sum of (25) and (27), we obtain

$$p\lambda_0\widehat{x}^{p-q}(t)+r\lambda c_r(t)\widehat{x}^{r-q}(t)=q|\psi(t)|^q.$$

It means that $\widehat{x}(\cdot)$ satisfies (4) for $\lambda_1=\ldots=\lambda_n=\lambda$.

Now we show that for

$$\lambda_0=\frac{q}{p}I_1^{\frac{p-q}{p}}\xi^{-\eta q-d\frac{p-q}{p}}\delta^{q-p} \tag{28}$$

the equalities

$$\int_T \widehat{x}^p(t)\,d\mu(t)=\delta^p,\quad \int_T |\varphi_j(t)|^r\widehat{x}^r(t)\,d\mu(t)=1,\ j=1,\ldots,n,$$

hold. In view of the definition of $\widehat{x}(\cdot)$ we need to check that

$$\int_T\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{p}{p-q}}k^{\frac{p}{p-q}}(\xi t)\,d\mu(t)=\delta^p,$$

$$\int_T|\varphi_j(t)|^r\left(\frac{q|\psi(t)|^q}{p\lambda_0}\right)^{\frac{r}{p-q}}k^{\frac{r}{p-q}}(\xi t)\,d\mu(t)=1,\ j=1,\ldots,n.$$

Changing $z=\xi t$ and taking into account that functions $|\psi(\cdot)|$, $|\varphi_j(\cdot)|$, $j=1,\ldots,n$, with the measure $\mu(\cdot)$ are homogenous, we obtain

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{p}{p-q}}I_1=\delta^p\xi^{\frac{\eta qp}{p-q}+d},\quad \left(\frac{q}{p\lambda_0}\right)^{\frac{r}{p-q}}I_{j+1}=\xi^{\frac{\eta qr}{p-q}+\nu r+d},\ j=1,\ldots,n.$$

The validity of these equalities immediately follows from the definitions of λ_0 and ξ.

It follows by Theorem 1, (28), (26), and (24) that

$$\begin{aligned}E^q(p,q,r)=\frac{p\lambda_0\delta^p+nr\lambda}{q}&=I_1^{\frac{p-q}{p}}\xi^{-\eta q-d\frac{p-q}{p}}\delta^q\\&+n\left(\frac{p\lambda_1}{q}\right)^{\frac{r-q}{p-q}}\xi^{\nu r-\eta\frac{q(p-r)}{p-q}}=\delta^{q\gamma}I_1^{-q\gamma/p}I_2^{-q(1-\gamma)/r}(I_1+nI_2).\end{aligned}$$

Moreover, the same theorem states that the method

$$\widehat{m}(y)(t)=q^{-1}p\lambda_0\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t)=k(\xi t)\psi(t)y(t)$$

is optimal.

2. Let $(p,q,r)\in P_1$. We use Theorem 2. Consider the function $\widehat{x}(\cdot)$ defined by (12) with $\lambda_1=\ldots=\lambda_n=\lambda$. Let us find λ_0 and λ from the conditions

$$\int_T \widehat{x}^p(t)\,d\mu(t)=\delta^p,\quad \int_T |\varphi_j(t)|^q\widehat{x}^q(t)\,d\mu(t)=1,\ j=1,\ldots,n.$$

Then we obtain

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{p}{p-q}}\int_T\left(|\psi(t)|^q-\lambda s_q(t)\right)_+^{\frac{p}{p-q}}\,d\mu(t)=\delta^p,$$

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{q}{p-q}}\int_T|\varphi_j(t)|^q\left(|\psi(t)|^q-\lambda s_q(t)\right)_+^{\frac{q}{p-q}}\,d\mu(t)=1,\ j=1,\ldots,n.$$

Put $\lambda=a^{(\eta-\nu)q}$, $a>0$. Changing $t=az$, we obtain

$$\left(\frac{q}{p\lambda_0}\right)^{\frac{p}{p-q}}a^{d+\frac{pq\eta}{p-q}}I_1=\delta^p,\quad \left(\frac{q}{p\lambda_0}\right)^{\frac{q}{p-q}}a^{d+q\nu+\frac{q^2\eta}{p-q}}I_{j+1}=1,\ j=1,\ldots,n.$$

It is easy to check that these equalities are fulfilled for

$$a=(I_1^{1/p}I_2^{-1/q}\delta^{-1})^{\frac{1}{\nu+d(1/q-1/p)}},\quad \lambda_0=\frac{q}{p}I_1I_2^{-1}\delta^{-p}(I_1^{-q/p}I_2\delta^q)^{\frac{\eta-\nu}{\nu+d(1/q-1/p)}}.$$

Substituting these values in (13) and (14) we obtain the statement of the theorem in the case under consideration.

3. Let $(p,q,r)\in P_2$. Here we use Theorem 3. Put $\lambda_1=\ldots=\lambda_n=\lambda$ in the definition of $\widehat{x}(\cdot)$ (see (16)). We find λ_0 and λ from the conditions

$$\int_T \widehat{x}^p(t)\,d\mu(t)=\delta^p,\qquad \int_T |\varphi_j(t)|^r\widehat{x}^r(t)\,d\mu(t)=1,\ j=1,\ldots,n.$$

We have

$$\left(\frac{p}{r\lambda}\right)^{\frac{p}{r-p}}\int_T\left(s_r^{-1}(t)(|\psi(t)|^p-\lambda_0)_+\right)^{\frac{p}{r-p}}\,d\mu(t)=\delta^p,$$

$$\left(\frac{p}{r\lambda}\right)^{\frac{r}{r-p}}\int_T|\varphi_j(t)|^r\left(s_r^{-1}(t)(|\psi(t)|^p-\lambda_0)_+\right)^{\frac{r}{r-p}}\,d\mu(t)=1,\ j=1,\ldots,n.$$

Put $\lambda_0=a^{\eta p}$, $a>0$. Changing $t=az$, we obtain

$$\left(\frac{p}{r\lambda}\right)^{\frac{p}{r-p}}a^{d+\frac{p^2\eta}{r-p}-\frac{pr\nu}{r-p}}I_1=\delta^p,$$

$$\left(\frac{p}{r\lambda}\right)^{\frac{r}{r-p}}a^{d+r\nu+\frac{pr\eta}{r-p}-\frac{r^2\nu}{r-p}}I_{j+1}=1,\ j=1,\ldots,n.$$

These equalities are valid for

$$a=(I_1^{1/p}I_2^{-1/r}\delta^{-1})^{\frac{1}{\nu+d(1/r-1/p)}},$$

$$\lambda=\frac{p}{r}I_1^{r/p-1}\delta^{p-r}(I_1^{r/p}I_2^{-1}\delta^{-r})^{\frac{p\eta/r-\nu-d(1/r-1/p)}{\nu+d(1/r-1/p)}}.$$

It remains to substitute these values into (17) and (18). □

Corollary 1. *Assume that conditions of Theorem 4 hold. Then for all $x(\cdot)\ne 0$ such that $x(\cdot)\in L_p(T,\mu)$ and $\varphi_j(\cdot)x(\cdot)\in L_r(T,\mu)$, $j=1,\ldots,n$, the sharp inequality*

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le C\|x(\cdot)\|^\gamma_{L_p(T,\mu)}\left(\max_{1\le j\le n}\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\right)^{1-\gamma} \tag{29}$$

holds, where

$$C=I_1^{-\gamma/p}I_2^{-(1-\gamma)/r}(I_1+nI_2)^{1/q}.$$

Proof. Let $x(\cdot)\in L_p(T,\mu)$, $\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}<\infty$, $j=1,\ldots,n$ and $x(\cdot)\ne 0$. Put

$$A=\max_{1\le j\le n}\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}.$$

Consider $\widehat{x}(\cdot)=x(\cdot)/A$. Put $\delta=\|\widehat{x}(\cdot)\|_{L_p(T_0,\mu)}$. Then $\|\varphi_j(\cdot)\widehat{x}(\cdot)\|_{L_r(T,\mu)}\le 1$, $j=1,\ldots,n$. In view of (20) and Theorem 4 we have

$$\|\psi(\cdot)\widehat{x}(\cdot)\|_{L_q(T,\mu)}\le C\|\widehat{x}(\cdot)\|^\gamma_{L_p(T,\mu)}.$$

This implies (29).

If there exists a $\widetilde{C}<C$ for which (29) holds, then

$$E(p,q,r)=\sup_{\substack{\|x(\cdot)\|_{L_p(T,\mu)}\le\delta\\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\ldots,n}}\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le\widetilde{C}\delta^\gamma<C\delta^\gamma.$$

This contradicts with (22). □

Let $|w(\cdot)|$, $|w_0(\cdot)|$ be homogenous functions of degrees θ, θ_0, respectively and $|w_j(\cdot)|$, $j=1,\ldots,n$, be homogenous functions of degree θ_1. We assume that $w(t), w_0(t)\neq 0$ and $\sum_{j=1}^n |w_j(t)|\neq 0$ for almost all $t\in T$.

For $(p,q,r)\in P$ we define $\widetilde{k}(\cdot)$ by the equality

$$\frac{\widetilde{k}^{\frac{1}{p-q}}(t)}{(1-\widetilde{k}(t))^{\frac{1}{r-q}}}=\left|\frac{w_0(t)}{w(t)}\right|^{\frac{p}{p-q}}\left(\sum_{j=1}^n\left|\frac{w_j(t)}{w(t)}\right|^r\right)^{-\frac{1}{r-q}}.$$

For $(p,q,r)\in P_1$ set

$$\widetilde{k}(t)=\left(1-|w(t)|^{-q}\sum_{j=1}^n|w_j(t)|^q\right)_+.$$

Put

$$\widetilde{\theta}=\theta+d/q,\quad \widetilde{\theta}_0=\theta_0+d/p,\quad \widetilde{\theta}_1=\theta_1+d/r,\quad \widetilde{\gamma}=\frac{\widetilde{\theta}_1-\widetilde{\theta}}{\widetilde{\theta}_1-\widetilde{\theta}_0}. \tag{30}$$

Corollary 2. *Let $(p,q,r)\in P\cup P_1\cup P_2$ and $\widetilde{\theta}_0\neq\widetilde{\theta}_1$. Assume that for $(p,q,r)\in P\cup P_1$*

$$\widetilde{I}_1=\int_T\left|\frac{w(t)}{w_0(t)}\right|^{\frac{qp}{p-q}}\widetilde{k}^{\frac{p}{p-q}}(t)\,d\mu(t)<\infty,$$

$$\widetilde{I}_{j+1}=\int_T\frac{|w(t)|^{\frac{qr}{p-q}}}{|w_0(t)|^{\frac{pr}{p-q}}}|w_j(t)|^r\widetilde{k}^{\frac{r}{p-q}}(t)\,d\mu(t)<\infty,\ j=1,\ldots,n,$$

and for $(p,q,r)\in P_2$

$$\widetilde{I}_1=\int_T|w_0(t)|^p\left(\frac{(|w(t)|^p-|w_0(t)|^p)_+}{\sum_{k=1}^n|w_k(t)|^r}\right)^{\frac{p}{r-p}}d\mu(t)<\infty,$$

$$\widetilde{I}_{j+1}=\int_T|w_j(t)|^r\left(\frac{(|w(t)|^p-|w_0(t)|^p)_+}{\sum_{k=1}^n|w_k(t)|^r}\right)^{\frac{r}{r-p}}d\mu(t)<\infty,\ j=1,\ldots,n.$$

Moreover, assume that $\widetilde{I}_2=\ldots=\widetilde{I}_{n+1}$. Then for all $x(\cdot)\neq 0$ such that $w_0(\cdot)x(\cdot)\in L_p(T,\mu)$ and $w_j(\cdot)x(\cdot)\in L_r(T,\mu)$, $j=1,\ldots,n$, the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le\widetilde{C}\|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu)}\left(\max_{1\le j\le n}\|\omega_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\right)^{1-\widetilde{\gamma}} \tag{31}$$

holds, where

$$\widetilde{C}=\widetilde{I}_1^{-\widetilde{\gamma}/p}\widetilde{I}_2^{-(1-\widetilde{\gamma})/r}(\widetilde{I}_1+n\widetilde{I}_2)^{1/q}.$$

Proof. Set

$$\psi(t)=\frac{w(t)}{w_0(t)},\quad \varphi_j(t)=\frac{w_j(t)}{w_0(t)},\ j=1,\ldots,n.$$

Then $|\psi(\cdot)|$ is a homogenous function of degree $\eta=\theta-\theta_0$ and $|\varphi_j(\cdot)|$, $j=1,\ldots,n$, are homogenous functions of degrees $\nu=\theta_1-\theta_0$. The quantity γ which was defined by (23) has the following form:

$$\widetilde{\gamma}=\frac{\widetilde{\theta}_1-\widetilde{\theta}}{\widetilde{\theta}_1-\widetilde{\theta}_0}.$$

It follows by Corollary 1 that for all $y(\cdot) \neq 0$ such that $y(\cdot) \in L_p(T,\mu)$ and $\varphi_j(\cdot)y(\cdot) \in L_r(T,\mu)$, $j = 1,\dots,n$, the sharp inequality

$$\|\psi(\cdot)y(\cdot)\|_{L_q(T,\mu)} \le \widetilde{C}\|y(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu)} \left(\max_{1\le j\le n} \|\varphi_j(\cdot)y(\cdot)\|_{L_r(T,\mu)}\right)^{1-\widetilde{\gamma}}.$$

holds. Substituting $y(\cdot) = w_0(\cdot)x(\cdot)$, we obtain (31). □

5. Homogenous weights in $\mathbb{R}^d$

Let T be a cone in $\mathbb{R}^d$, $d\mu(t) = dt$, $|\psi(\cdot)|$ be homogenous function of degree η, $|\varphi_j(\cdot)|$, $j = 1,\dots,n$, be homogenous functions of degrees ν, $\psi(t) \neq 0$ and $\sum_{j=1}^n |\varphi_j(t)| \neq 0$ for almost all $t \in T$. Consider the polar transformation

$$\begin{aligned} t_1 &= \rho\cos\omega_1,\\ t_2 &= \rho\sin\omega_1\cos\omega_2,\\ &\dots\dots\dots\dots\dots\dots\\ t_{d-1} &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\cos\omega_{d-1},\\ t_d &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1}. \end{aligned}$$

Set $\omega = (\omega_1,\dots,\omega_{d-1})$. For any function $f(\cdot)$ we put

$$\widetilde{f}(\omega) = |f(\cos\omega_1,\dots,\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|. \tag{32}$$

Note that if $|f(\cdot)|$ is a homogenous function of degree κ, then $f(\omega) = \rho^{-\kappa}|f(t)|$. Denote by Ω the range of ω. Since T is a cone, Ω does not depend on ρ. Put

$$J(\omega) = \sin^{d-2}\omega_1\sin^{d-3}\omega_2\dots\sin\omega_{d-2}.$$

Assume that $\gamma \in (0,1)$, where γ is defined by (23). Put

$$\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}. \tag{33}$$

It is easy to verify that $q^* > q \ge 1$. Moreover,

$$q^* = \frac{pqr(\nu + d(1/r - 1/p))}{\nu r(p-q) - \eta q(p-r)}.$$

Theorem 5. *Let $(p,q,r) \in P \cup P_1 \cup P_2$ and $\gamma \in (0,1)$. Assume that*

$$I = \int_\Omega \frac{\widetilde{\psi}^{q^*}(\omega)}{\widetilde{s}_r^{q^*(1-\gamma)/r}(\omega)} J(\omega)\,d\omega < \infty,$$

and $I_1' = \dots = I_n'$, where

$$I_j' = \int_\Omega \frac{\widetilde{\psi}^{q^*}(\omega)\widetilde{\varphi}_j^r(\omega)}{\widetilde{s}_r^{q^*(1-\gamma)/r+1}(\omega)} J(\omega)\,d\omega,\ \ j = 1,\dots,n.$$

Then

$$E(p,q,r) = K\delta^\gamma, \tag{34}$$

where

$$K = \gamma^{-\frac{\gamma}{p}}\left(\frac{1-\gamma}{n}\right)^{-\frac{1-\gamma}{r}}\left(\frac{B\left(q^*\gamma/p, q^*(1-\gamma)/r\right)I}{|\nu + d(1/r - 1/p)|(\gamma r + (1-\gamma)p)}\right)^{1/q^*},$$

where $B(\cdot,\cdot)$ is the Eiler beta-function. Moreover, the method

$$\widehat{m}(y)(t) = \kappa\left(\widehat{\xi}^{\frac{1}{\nu + d(1/r - 1/p)}} t\right)\psi(t)y(t),$$

where

$$\widehat{\xi} = \delta\gamma^{-1/p}\left(\frac{1-\gamma}{n}\right)^{1/r}\left(\frac{B\left(q^*\gamma/p, q^*(1-\gamma)/r\right)I}{|\nu + d(1/r-1/p)|(\gamma r + (1-\gamma)p)}\right)^{1/r-1/p},$$

is optimal recovery method.

Proof. First of all, we note that $I_1' + \ldots + I_n' = I$. Consequently, $I_j' = I/n$, $j = 1, \ldots, n$. We will apply Theorem 4.

1. Let $(p, q, r) \in P$. Passing to the polar transformation we obtain

$$\frac{k^{\frac{1}{p-q}}(\rho,\omega)}{(1-k(\rho,\omega))^{\frac{1}{r-q}}} = \rho^{\frac{\eta q(p-r)-\nu r(p-q)}{(p-q)(r-q)}} \frac{\widetilde{\psi}^{\frac{q(p-r)}{(p-q)(r-q)}}(\omega)}{\widetilde{s}_r^{\frac{1}{r-q}}(\omega)}.$$

Using the same scheme of calculation of I_1 as it was given in [8, Theorem 3], we obtain

$$I_1 = \frac{\gamma}{pr|\nu + d(1/r-1/p)|}\left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q})I,$$

where

$$\widehat{p} = q^*\frac{\gamma}{p}, \quad \widehat{q} = q^*\frac{1-\gamma}{r}.$$

In a similar way we calculate

$$I_{j+1} = \frac{1-\gamma}{pr|\nu + d(1/r-1/p)|}\left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q})I_j', \ j = 1, \ldots, n.$$

Thus,

$$I_2 = \frac{1-\gamma}{npr|\nu + d(1/r-1/p)|}\left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q})I.$$

It remains to substitute these values into (22) and (24).

2. Let $(p, q, r) \in P_1$. Now we use the scheme of calculation of I_1 which was given in [10, Theorem 3]. We obtain

$$\begin{aligned} I_1 &= \frac{I}{|\nu-\eta|q} B\left(q^*\gamma/p + 2, q^*(1-\gamma)/q\right) \\ &= \frac{I}{|\nu-\eta|q}\frac{q^*\gamma/p+1}{q^*\gamma/p + 1 + q^*(1-\gamma)/q} B\left(q^*\gamma/p + 1, q^*(1-\gamma)/q\right). \end{aligned}$$

Since $r = q$ we have

$$\frac{1}{q^*} = \gamma\left(\frac{1}{q} - \frac{1}{p}\right), \quad \gamma = \frac{\nu - \eta}{\nu + d(1/q - 1/p)}.$$

Therefore, $q^*\gamma/p + 1 = q^*\gamma/q$. Hence

$$\begin{aligned} I_1 &= \frac{I\gamma}{|\nu-\eta|q} B\left(q^*\gamma/p + 1, q^*(1-\gamma)/q\right) \\ &= \frac{I\gamma}{|\nu-\eta|q}\frac{q^*\gamma/p}{q^*\gamma/p + q^*(1-\gamma)/q} B\left(q^*\gamma/p, q^*(1-\gamma)/q\right) \\ &= \frac{\gamma}{pr|\nu + d(1/r-1/p)|}\left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q})I. \end{aligned}$$

By the similar way we get

$$\begin{aligned}
I_{j+1} &= \frac{I_j'}{|\nu-\eta|q}B\left(q^*\gamma/p+1, q^*(1-\gamma)/q+1\right)\\
&= \frac{I_j'}{|\nu-\eta|q}\frac{q^*\gamma/p}{q^*\gamma/p+q^*(1-\gamma)/q+1}B\left(q^*\gamma/p, q^*(1-\gamma)/q+1\right)\\
&= \frac{I_j'\gamma}{|\nu-\eta|p}B\left(q^*\gamma/p, q^*(1-\gamma)/q+1\right) = \frac{I_j'\gamma}{|\nu-\eta|p}\frac{q^*(1-\gamma)/q}{q^*\gamma/p+q^*(1-\gamma)/q}B\left(q^*\gamma/p, q^*(1-\gamma)/q\right)\\
&= \frac{1-\gamma}{npr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I.
\end{aligned}$$

Thus, we obtain the same formulas for I_1 and I_2 as in the first case.

3. Let $(p,q,r)\in P_2$. Here we use the scheme of calculation of J_1 and J_2 which was given in [10, Theorem 3]. We obtain

$$\begin{aligned}
I_1 &= \frac{I}{|\eta|p}B\left(q^*\gamma/p+1, q^*(1-\gamma)/r+1\right),\\
I_{j+1} &= \frac{I_j'}{|\eta|p}B\left(q^*\gamma/p, q^*(1-\gamma)/r+2\right),\ j=1,\dots,n.
\end{aligned}$$

Since $q=p$ we have

$$\frac{1}{q^*}=(1-\gamma)\left(\frac{1}{p}-\frac{1}{r}\right),\quad 1-\gamma=\frac{\eta}{\nu+d(1/r-1/p)}.$$

Therefore, $q^*(1-\gamma)/r+1=q^*(1-\gamma)/p$. Hence

$$\begin{aligned}
I_1 &= \frac{I}{|\eta|p}\frac{q^*\gamma/p}{q^*\gamma/p+q^*(1-\gamma)/r+1}B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right)\\
&= \frac{I\gamma}{|\eta|p}B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right) = \frac{I\gamma}{|\eta|p}\frac{q^*(1-\gamma)/r}{q^*\gamma/p+q^*(1-\gamma)/r}B\left(q^*\gamma/p, q^*(1-\gamma)/r\right)\\
&= \frac{\gamma}{pr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I.
\end{aligned}$$

For I_{j+1}, $j=1,\dots,n$, we have

$$\begin{aligned}
I_{j+1} &= \frac{I_j'}{|\eta|p}\frac{q^*(1-\gamma)/r+1}{q^*\gamma/p+q^*(1-\gamma)/r+1}B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right)\\
&= \frac{I_j'(1-\gamma)}{|\eta|p}B\left(q^*\gamma/p, q^*(1-\gamma)/r+1\right) = \frac{I_j'(1-\gamma)}{pr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})\\
&= \frac{1-\gamma}{npr|\nu+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I.
\end{aligned}$$

Again we obtain the same formulas for I_1 and I_2 as in the previous cases. □

For $n=1$ Theorem 5 was proved in [10]. Analogously to Corollary 1 we obtain

Corollary 3. *Assume that conditions of Theorem 5 hold. Then for all $x(\cdot)$ such that $x(\cdot)\in L_p(T,\mu)$ and $\varphi_j(\cdot)x(\cdot)\in L_r(T,\mu)$, $j=1,\dots,n$, the sharp inequality*

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le K\|x(\cdot)\|^{\gamma}_{L_p(T,\mu)}\left(\max_{1\le j\le n}\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\right)^{1-\gamma}$$

holds.

Let $|w(\cdot)|$, $|w_0(\cdot)|$ be homogenous functions of degrees θ, θ_0, respectively and $|w_j(\cdot)|$, $j=1,\ldots,n$, be homogenous functions of degree θ_1. We assume that $w(t), w_0(t) \neq 0$ and $\sum_{j=1}^n |w_j(t)| \neq 0$ for almost all $t \in T$. Define $\widetilde{w}(\cdot)$, $\widetilde{w}_0(\cdot)$, $\widetilde{w}_1(\cdot)$ by (32). Similar to Corollary 2 we obtain

Corollary 4. *Let $(p,q,r) \in P \cup P_1 \cup P_2$ and $\widetilde{\gamma} \in (0,1)$ where $\widetilde{\gamma}$ is defined by (30). Assume that*

$$\widetilde{I} = \int_\Omega \frac{\widetilde{w}^{\widetilde{q}}(\omega)}{\widetilde{w}_0^{\widetilde{q}\widetilde{\gamma}}(\omega)\left(\sum_{k=1}^n \widetilde{w}_k^r(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r}} J(\omega)\, d\omega < \infty,$$

where

$$\frac{1}{\widetilde{q}} = \frac{1}{q} - \frac{\widetilde{\gamma}}{p} - \frac{1-\widetilde{\gamma}}{r},$$

and $\widetilde{I}'_1 = \ldots = \widetilde{I}'_n$, where

$$\widetilde{I}'_j = \int_\Omega \frac{\widetilde{w}^{\widetilde{q}}(\omega)\widetilde{w}_j^r(\omega)}{\widetilde{w}_0^{\widetilde{q}\widetilde{\gamma}}(\omega)\left(\sum_{k=1}^n \widetilde{w}_k^r(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r+1}} J(\omega)\, d\omega,\ j=1,\ldots,n.$$

Then for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot) \in L_p(T,\mu)$ and $w_j(\cdot)x(\cdot) \in L_r(T,\mu)$, $j=1,\ldots,n$, the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \leq \widetilde{K} \|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu))} \left(\max_{1\leq j\leq n} \|\omega_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\right)^{1-\widetilde{\gamma}}$$

holds, where

$$\widetilde{K} = \widetilde{\gamma}^{-\frac{\widetilde{\gamma}}{p}} \left(\frac{1-\widetilde{\gamma}}{n}\right)^{-\frac{1-\widetilde{\gamma}}{r}} \left(\frac{B\left(\widetilde{q}\widetilde{\gamma}/p, \widetilde{q}(1-\widetilde{\gamma})/r\right)\widetilde{I}}{|\widetilde{\theta}_1 - \widetilde{\theta}_0|(\widetilde{\gamma}r + (1-\widetilde{\gamma})p)}\right)^{1/\widetilde{q}}. \tag{35}$$

The statement of Corollary 4 for $(p,q,r) \in P$ and $n=1$ was proved in [2].

We give an example of weights for which conditions of Corollary 4 hold. Let $T = \mathbb{R}^d_+$, $\theta_1 > 0$,

$$w(t) = (t_1^2 + \ldots + t_d^2)^{\theta/2}, \quad w_0(t) = (t_1^2 + \ldots + t_d^2)^{\theta_0/2}, \quad w_j(t) = t_j^{\theta_1},\ j=1,\ldots,d. \tag{36}$$

The condition $0 < \widetilde{\gamma} < 1$ is equivalent to inequalities $\widetilde{\theta}_1 > \widetilde{\theta} > \widetilde{\theta}_0$ or $\widetilde{\theta}_1 < \widetilde{\theta} < \widetilde{\theta}_0$. Therefore, we assume that for θ and θ_0 inequalities $\theta_1 + d(1/r - 1/q) > \theta > \theta_0 + d(1/p - 1/q)$ or $\theta_1 + d(1/r - 1/q) < \theta < \theta_0 + d(1/p - 1/q)$ hold.

It is easy to check that $\widetilde{w}(\cdot) = \widetilde{w}_0(\cdot) = 1$ and $\widetilde{w}_j(\omega) = \widetilde{t}_j^{\theta_1}(\omega)$, $j=1,\ldots,d$, where

$$\begin{aligned}
\widetilde{t}_1(\omega) &= \cos\omega_1,\\
\widetilde{t}_2(\omega) &= \sin\omega_1 \cos\omega_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\widetilde{t}_{d-1}(\omega) &= \sin\omega_1 \sin\omega_2 \ldots \sin\omega_{d-2}\cos\omega_{d-1},\\
\widetilde{t}_d(\omega) &= \sin\omega_1 \sin\omega_2 \ldots \sin\omega_{d-2}\sin\omega_{d-1}.
\end{aligned}$$

Note that

$$\sum_{k=1}^d t_k^2(\omega) = 1.$$

For $\widetilde{I}$ we have

$$\widetilde{I}=\int_{\Pi_+^{d-1}}\frac{J(\omega)\,d\omega}{\left(\sum_{k=1}^d\widetilde{t}_k^{r\theta_1}(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r}},\quad \Pi_+^{d-1}=[0,\pi/2]^{d-1}. \tag{37}$$

If $r\theta_1\le 2$, then

$$\sum_{k=1}^d\widetilde{t}_k^{r\theta_1}(\omega)\ge\sum_{k=1}^d\widetilde{t}_k^2(\omega)=1. \tag{38}$$

For $r\theta_1>2$ by Hölder's inequality

$$1=\sum_{k=1}^d\widetilde{t}_k^2(\omega)\le\left(\sum_{k=1}^d\widetilde{t}_k^{r\theta_1}(\omega)\right)^{\frac{2}{r\theta_1}}d^{1-\frac{2}{r\theta_1}}.$$

Thus,

$$\sum_{k=1}^d\widetilde{t}_k^{r\theta_1}(\omega)\ge d^{1-\frac{r\theta_1}{2}}. \tag{39}$$

It follows by (38) and (39) that $\widetilde{I}<\infty$.

For $\widetilde{I}_j'$ we have

$$\widetilde{I}_j'=\int_{\Pi_+^{d-1}}\frac{\widetilde{t}_j^{r\theta_1}J(\omega)\,d\omega}{\left(\sum_{k=1}^d\widetilde{t}_k^{r\theta_1}(\omega)\right)^{\widetilde{q}(1-\widetilde{\gamma})/r+1}},\ j=1,\dots,d.$$

Consider the integrals

$$L_j=\int_{\mathbb{R}_+^d\cap\mathbb{B}^d}\frac{\left(\sum_{k=1}^d t_k^2\right)^{\theta_1\widetilde{q}(1-\widetilde{\gamma})/2}t_j^{r\theta_1}}{\left(\sum_{k=1}^d t_k^{r\theta_1}\right)^{\widetilde{q}(1-\widetilde{\gamma})/r+1}}\,dt,\ j=1,\dots,d,$$

where $\mathbb{B}^d$ is the unit ball in $\mathbb{R}^d$. If we change variables in L_j changing places variables t_j and t_k, then L_j passes to L_k. Therefore, $L_1=\ldots=L_d$. Passing to the polar transformation we obtain that $L_j=\widetilde{I}_j'/d$, $j=1,\dots,d$. Consequently, $\widetilde{I}_1'=\ldots=\widetilde{I}_d'$.

Thus, we obtain

Corollary 5. *Let $(p,q,r)\in P\cup P_1\cup P_2$, $\theta_1>0$, θ and θ_0 be such that $\theta_1+d(1/r-1/q)>\theta>\theta_0+d(1/p-1/q)$ or $\theta_1+d(1/r-1/q)<\theta<\theta_0+d(1/p-1/q)$. Then for weights* (36) *and all $x(\cdot)$ for which $w_0(\cdot)x(\cdot)\in L_p(\mathbb{R}_+^d)$ and $w_j(\cdot)x(\cdot)\in L_r(\mathbb{R}_+^d)$, $j=1,\dots,d$, the sharp inequality*

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}_+^d)}\le\widetilde{K}\|w_0(\cdot)x(\cdot)\|_{L_p(\mathbb{R}_+^d)}^{\widetilde{\gamma}}\left(\max_{1\le j\le d}\|\omega_j(\cdot)x(\cdot)\|_{L_r(\mathbb{R}_+^d)}\right)^{1-\widetilde{\gamma}}$$

holds, where $\widetilde{K}$ is defined by (35) *in which the value $\widetilde{I}$ is defined by* (37).

We give one more example.

Corollary 6. *Let $(p,q,r)\in P\cup P_1\cup P_2$, weights $w(\cdot)$, $w_0(\cdot)$, $w_1(\cdot)$ be defined by* (36) *for $\theta=d(1-1/q)$, $\theta_0=d-(\lambda+d)/p$, $\theta_1=d+(\mu-d)/r$, where $\lambda,\mu>0$. Put*

$$\alpha=\frac{\mu}{p\mu+r\lambda},\quad\beta=\frac{\lambda}{p\mu+r\lambda}.$$

Then for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot) \in L_p(\mathbb{R}^d_+)$ and $w_j(\cdot)x(\cdot) \in L_r(\mathbb{R}^d_+)$, $j = 1, \dots, d$, the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d_+)} \le C\|w_0(\cdot)x(\cdot)\|^{p\alpha}_{L_p(\mathbb{R}^d_+)} \left(\max_{1\le j\le d} \|\omega_j(\cdot)x(\cdot)\|_{L_r(\mathbb{R}^d_+)}\right)^{r\beta}$$

holds, where

$$C = \frac{d^\beta}{(p\alpha)^\alpha (r\beta)^\beta} \left(\frac{I}{\lambda+\mu} B\left(\frac{\alpha}{1/q-\alpha-\beta}, \frac{\beta}{1/q-\alpha-\beta}\right)\right)^{1/q-\alpha-\beta},$$

and

$$I = \int_{\Pi^{d-1}_+} \frac{J(\omega)\, d\omega}{\left(\sum_{k=1}^d \widetilde{t}_k^{\,r(d-1)+\mu}(\omega)\right)^{\frac{\beta}{1/q-\alpha-\beta}}}.$$

For $d = 1$, $q = 1$, and $(p, 1, r) \in P$ the statement of Corollary 6 was proved in [4].

6. Recovery of differential operators from a noisy Fourier transform

Let T be a cone in $\mathbb{R}^d$, $d\mu(t) = dt$, $|\psi(\cdot)|$ be homogenous function of degree η, $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, be homogenous functions of degrees ν, $\psi(t) \ne 0$ and $\sum_{j=1}^n |\varphi_j(t)| \ne 0$ for almost all $t \in T$.

Let S be the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$, S' be the corresponding space of distributions, and let $F\colon S' \to S'$ be the Fourier transform. Set

$$X_p = \left\{ x(\cdot) \in S' : \varphi_j(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d),\ j = 1, \dots, n,\ Fx(\cdot) \in L_p(\mathbb{R}^d) \right\}.$$

We define operators D_j, $j = 1, \dots, n$, as follows

$$D_j x(\cdot) = F^{-1}(\varphi_j(\cdot)Fx(\cdot))(\cdot),\ j = 1, \dots, n.$$

Put

$$\Lambda x(\cdot) = F^{-1}(\psi(\cdot)Fx(\cdot))(\cdot). \tag{40}$$

Consider the problem of the optimal recovery of values of the operator Λ on the class

$$W_p^{\mathcal{D}} = \left\{ x(\cdot) \in X_p : \|D_j x(\cdot)\|_{L_2(\mathbb{R}^d)} \le 1,\ j = 1, \dots, n \right\}, \quad \mathcal{D} = (D_1, \dots, D_n),$$

from the noisy Fourier transform of the function $x(\cdot)$. We assume that for each $x(\cdot) \in W_p$ one knows a function $y(\cdot) \in L_p(\mathbb{R}^d)$ such that $\|Fx(\cdot) - y(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta$, $\delta > 0$. It is required to recover the function $\Lambda x(\cdot)$ from $y(\cdot)$. Assume that $\Lambda x(\cdot) \in L_q(\mathbb{R}^d)$ for all $x(\cdot) \in X_p$. As recovery methods we consider all possible mappings $m\colon L_p(\mathbb{R}^d) \to L_q(\mathbb{R}^d)$. The error of a method m is defined by

$$e_{pq}(\Lambda, \mathcal{D}, m) = \sup_{\substack{x(\cdot)\in W_p^{\mathcal{D}},\ y(\cdot)\in L_p(\mathbb{R}^d) \\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta}} \|\Lambda x(\cdot) - m(y)(\cdot)\|_{L_q(\mathbb{R}^d)}.$$

The quantity

$$E_{pq}(\Lambda, \mathcal{D}) = \inf_{m\colon L_p(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} e_{pq}(\Lambda, \mathcal{D}, m) \tag{41}$$

is called the error of optimal recovery, and the method on which the infimum is attained, an optimal method.

1. **Recovery in the metric $L_2(\mathbb{R}^d)$.** By Plancherels theorem,

$$\|\Lambda x(\cdot)-m(y)(\cdot)\|_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^{d/2}}\|\widetilde{L}x(\cdot)-F(m(y))(\cdot)\|_{L_2(\mathbb{R}^d)},$$

where $\widetilde{L}x(\cdot)=\psi(\cdot)Fx(\cdot)$. Moreover,

$$\|D_jx(\cdot)\|_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^{d/2}}\|\varphi_j(\cdot)Fx(\cdot)\|_{L_2(\mathbb{R}^d)},\ j=1,\dots,n.$$

So, the problem under consideration coincides, up to a factor of $(2\pi)^{-d/2}$, with problem (2) for $q=r=2$ with $\varphi_j(\cdot)$ replaced by $(2\pi)^{-d/2}\varphi_j(\cdot)$, $j=1,\dots,n$.

For $q=r=2$ we denote by $\widehat{\gamma}$ and $\widehat{q}^*$ the values γ and q^*, which where defined by (23) and (33):

$$\widehat{\gamma}=\frac{\nu-\eta}{\nu+d(1/2-1/p)},\quad \widehat{q}^*=\frac{1}{\widehat{\gamma}(1/2-1/p)}.$$

Set

$$C_p(\nu,\eta)=\widehat{\gamma}^{-\frac{\widehat{\gamma}}{p}}\left(\frac{1-\widehat{\gamma}}{n}\right)^{-\frac{1-\widehat{\gamma}}{2}}\left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1,\widehat{q}^*(1-\widehat{\gamma})/2\right)}{2|\nu-\eta|}\right)^{1/\widehat{q}^*}.$$

Theorem 6. *Let $2<p\le\infty$, $\widehat{\gamma}\in(0,1)$. Assume that*

$$I=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}^*}(\omega)}{\widetilde{s}_2^{\widehat{q}^*(1-\widehat{\gamma})/2}(\omega)}J(\omega)\,d\omega<\infty,\quad \Pi^{d-1}=[0,\pi]^{d-2}\times[0,2\pi]\tag{42}$$

and $I_1'=\ldots=I_n'$, where

$$I_j'=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}^*}(\omega)\widetilde{\varphi}_j^2(\omega)}{\widetilde{s}_2^{\widehat{q}^*(1-\widehat{\gamma})/2+1}(\omega)}J(\omega)\,d\omega,\ j=1,\dots,n.\tag{43}$$

Then

$$E_{p2}(\Lambda,\mathcal{D})=\frac{1}{(2\pi)^{d\widehat{\gamma}/2}}C_p(\nu,\eta)I^{1/\widehat{q}^*}\delta^{\widehat{\gamma}}.\tag{44}$$

The method

$$\widehat{m}(y)(t)=F^{-1}\left(\left(1-\beta\frac{s_2(t)}{|\psi(t)|^2}\right)_+\psi(t)y(t)\right),\tag{45}$$

where

$$\beta=\frac{1-\widehat{\gamma}}{n(2\pi)^{d\widehat{\gamma}}}C_p^2(\nu,\eta)\left(\delta I^{1/2-1/p}\right)^{2\widehat{\gamma}},$$

is optimal.

Moreover, the sharp inequality

$$\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\frac{1}{(2\pi)^{d\widehat{\gamma}/2}}C_p(\nu,\eta)I^{1/\widehat{q}^*}\|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\widehat{\gamma}}\left(\max_{1\le j\le n}\|D_jx(\cdot)\|_{L_2(\mathbb{R}^d)}\right)^{1-\widehat{\gamma}}\tag{46}$$

holds.

Proof. Let $2 < p < \infty$. By Theorem 5 we have

$$E_{p2}(\Lambda,\mathcal{D}) = \frac{1}{(2\pi)^{d\widehat{\gamma}/2}} K\delta^{\widehat{\gamma}},$$

where

$$K = \widehat{\gamma}^{-\frac{\widehat{\gamma}}{p}} \left(\frac{1-\widehat{\gamma}}{n}\right)^{-\frac{1-\widehat{\gamma}}{2}} \left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p, \widehat{q}^*(1-\widehat{\gamma})/2\right) I}{|\nu + d(1/2-1/p)|(2\widehat{\gamma} + (1-\widehat{\gamma})p)}\right)^{1/\widehat{q}^*}.$$

From the properties of the beta-function we find that

$$(47)\quad \frac{B\left(\widehat{q}^*\widehat{\gamma}/p, \widehat{q}^*(1-\widehat{\gamma})/2\right)}{|\nu + d(1/2-1/p)|(2\widehat{\gamma} + (1-\widehat{\gamma})p)} = \frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1, \widehat{q}^*(1-\widehat{\gamma})/2\right)\left(\widehat{q}^*\widehat{\gamma}/p + \widehat{q}^*(1-\widehat{\gamma})/2\right)}{|\nu + d(1/2-1/p)|(2\widehat{\gamma} + (1-\widehat{\gamma})p)\widehat{q}^*\widehat{\gamma}/p} = \frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1, \widehat{q}^*(1-\widehat{\gamma})/2\right)}{2|\nu-\eta|}.$$

Thus, equality (44) holds.

It follows by Theorem 5 that the method

$$\widehat{m}(y)(t) = \left(1 - \frac{\widehat{\xi}^{2\widehat{\gamma}} c_2(t)}{(2\pi)^d|\psi(t)|^2}\right)_+ \psi(t)y(t),$$

where

$$\widehat{\xi} = \delta(2\pi)^{d\frac{1-\widehat{\gamma}}{2\widehat{\gamma}}} \widehat{\gamma}^{-1/p} \left(\frac{1-\widehat{\gamma}}{n}\right)^{1/2} \left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p, \widehat{q}^*(1-\widehat{\gamma})/2\right) I}{|\nu + d(1/2-1/p)|(2\widehat{\gamma} + (1-\widehat{\gamma})p)}\right)^{1/2-1/p},$$

is optimal. In view of (47) we obtain

$$\frac{\widehat{\xi}^{2\widehat{\gamma}}}{(2\pi)^d} = \frac{\delta^{2\widehat{\gamma}}\widehat{\gamma}^{-2\widehat{\gamma}/p}}{(2\pi)^{d\widehat{\gamma}}} \left(\frac{1-\widehat{\gamma}}{n}\right)^{\widehat{\gamma}} \left(\frac{B\left(\widehat{q}^*\widehat{\gamma}/p+1, \widehat{q}^*(1-\widehat{\gamma})/2\right) I}{2|\nu-\eta|}\right)^{2\widehat{\gamma}(1/2-1/p)} = \frac{1-\widehat{\gamma}}{n(2\pi)^{d\widehat{\gamma}}} C_p^2(\nu,\eta) \left(\delta I^{1/2-1/p}\right)^{2\widehat{\gamma}}.$$

Inequality (46) follows from Corollary 3. Consider the case $p = \infty$. It follows by Lemma 1 that

$$(48)\quad E_{\infty 2}(\Lambda,\mathcal{D}) \geq \sup_{\substack{x(\cdot)\in W_\infty^{\mathcal{D}} \\ \|Fx(\cdot)\|_{L_\infty(\mathbb{R}^d)}\leq\delta}} \|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi) = \begin{cases} \delta, & |\psi(\xi)| > \lambda\sqrt{s_2(\xi)}, \\ 0, & |\psi(\xi)| \leq \lambda\sqrt{s_2(\xi)}. \end{cases}$$

We show that $\lambda > 0$ may be selected from the condition

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 |F\widehat{x}(\xi)|^2\, d\xi = 1,\ j = 1,\dots,n.$$

Thus, $\lambda > 0$ should be chosen from the condition

$$\delta^2 \int_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}} |\varphi_j(\xi)|^2\, d\xi = (2\pi)^d.$$

Passing to the polar transformation for $\nu > \eta$ we obtain

$$\delta^2 \int_{\Pi_{d-1}} \widetilde{\varphi}_j^2(\omega) J(\omega)\, d\omega \int_0^{\Phi_1(\omega)} \rho^{2\nu+d-1}\, d\rho = (2\pi)^d, \quad \Phi_1(\omega) = \left(\frac{\widetilde{\psi}(\omega)}{\lambda\sqrt{\widetilde{s}_2(\xi)}}\right)^{\frac{1}{\nu-\eta}}.$$

If $\nu < \eta$, then $2\nu + d < 0$ (since $\widehat{\gamma} \in (0,1)$) and we have

$$\delta^2 \int_{\Pi_{d-1}} \widetilde{\varphi}_j^2(\omega) J(\omega)\, d\omega \int_{\Phi_1(\omega)}^{+\infty} \rho^{2\nu+d-1}\, d\rho = (2\pi)^d.$$

Hence

$$\frac{\delta^2}{|2\nu+d|} \lambda^{-\frac{2\nu+d}{\nu-\eta}} I_j' = (2\pi)^d.$$

As already noted, it follows from the equality $I_1' + \ldots + I_n' = I$ that $I_j' = I/n$, $j = 1, \ldots, n$. Consequently,

$$\lambda = \left(\frac{\delta^2 I}{(2\pi)^d n |2\nu+d|}\right)^{\frac{\nu-\eta}{2\nu+d}}.$$

It is easily checked that

$$C_\infty^2(\nu,\eta) = \frac{1}{|2\eta+d|} (n|2\nu+d|)^{\frac{\eta+d/2}{\nu+d/2}}.$$

As a result, $\lambda^2 = \beta$. In view of (48), using calculations similar to those that were above, we obtain

$$\text{(49)} \quad E_{\infty 2}^2(\Lambda, \mathcal{D}) \ge \|\Lambda\widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{\delta^2}{(2\pi)^d} \int_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}} |\psi(\xi)|^2\, d\xi$$
$$= \frac{\delta^2}{|2\eta+d|(2\pi)^d} \lambda^{-\frac{2\eta+d}{\nu-\eta}} I = \frac{1}{(2\pi)^{d\widehat{\gamma}}} C_\infty^2(\nu,\eta) I^{2/\widehat{q}^*} \delta^{2\widehat{\gamma}}.$$

We estimate the error of the method (45). Put

$$a(\xi) = \left(1 - \beta \frac{s_2(\xi)}{|\psi(\xi)|^2}\right)_+.$$

Taking the Fourier transform we obtain

$$\|\Lambda x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\psi(\xi)|^2 \left|Fx(\xi) - a(\xi)y(\xi)\right|^2\, d\xi.$$

We set $z(\cdot) = Fx(\cdot) - y(\cdot)$ and note that

$$\|z(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le \delta, \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 |Fx(\xi)|^2\, d\xi \le 1, \ j = 1, \ldots, n.$$

Hence

$$\|\Lambda x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\psi(\xi)|^2 \left|(1-a(\xi))\, Fx(\xi) + a(\xi)z(\xi)\right|^2\, d\xi.$$

The integrand can be written as

$$\left|\frac{|\psi(\xi)|(1-a(\xi))\sqrt{\beta s_2(\xi)} Fx(\xi)}{\sqrt{\beta s_2(\xi)}} + \sqrt{a(\xi)}\sqrt{a(\xi)}|\psi(\xi)|z(\xi)\right|^2.$$

Using the Cauchy-Bunyakovskii-Schwarz inequality

$$|ab + cd|^2 \le (|a|^2 + |c|^2)(|b|^2 + |d|^2)$$

we obtain the estimate

$$\|\Lambda x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}\le \operatorname*{vraisup}_{\xi\in\mathbb{R}^d} S(\xi)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\beta s_2(\xi)|Fx(\xi)|^2+a(\xi)|\psi(\xi)|^2|z(\xi)|^2\right)d\xi,$$

where

$$S(\xi)=\frac{|\psi(\xi)|^2(1-a(\xi))^2}{\beta s_2(\xi)}+a(\xi).$$

If $|\psi(\xi)|^2\le\beta s_2(\xi)$, then $a(\xi)=0$ and $S(\xi)\le 1$. If $|\psi(\xi)|^2>\beta s_2(\xi)$, then $S(\xi)=1$. So we have

$$e^2_{\infty 2}(\Lambda,\mathcal{D},\widehat{m})\le\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\beta s_2(\xi)|Fx(\xi)|^2+a(\xi)|\psi(\xi)|^2|z(\xi)|^2\right)d\xi\le n\beta$$
$$+\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}\left(|\psi(\xi)|^2-\beta s_2(\xi)\right)d\xi=n\beta+\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}|\psi(\xi)|^2\,d\xi$$
$$-\beta\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}s_2(\xi)|F\widehat{x}(\xi)|^2\,d\xi=\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}|\psi(\xi)|^2\,d\xi\le E^2_{\infty 2}(\Lambda,\mathcal{D}).$$

It follows that the method $\widehat{m}(y)(\cdot)$ is optimal. Moreover, by (49) we have

$$E^2_{\infty 2}(\Lambda,\mathcal{D})=\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda\sqrt{s_2(\xi)}}|\psi(\xi)|^2\,d\xi=\frac{1}{(2\pi)^{d\widehat{\gamma}}}C^2_\infty(n,k)I^{2/\widehat{q}^*}\delta^{2\widehat{\gamma}}.$$

Similar to the proof of Corollary 1 we prove that for $p=\infty$ inequality (46) is sharp . □

Let $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{R}^d_+$. We define the operator D^α (the derivative of order α) by

$$D^\alpha x(\cdot)=F^{-1}((i\xi)^\alpha Fx(\xi))(\cdot),$$

where $(i\xi)^\alpha=(i\xi_1)^{\alpha_1}\dots(i\xi_d)^{\alpha_d}$.

Consider problem (41) for $D_j=D^{\nu e_j}$, $j=1,\dots,d$, where e_j, $j=1\dots,d$, is a standard basis in $\mathbb{R}^d$, and Λ defined by (40). Assume that $\psi(\cdot)$ has the following symmetry property

$$\psi(\dots,\xi_j,\dots,\xi_m,\dots)=\psi(\dots,\xi_m,\dots,\xi_j,\dots),\quad 1\le j,m\le d.$$

Moreover, we assume that $\widetilde{\psi}(\cdot)$ is continuous function on Π^{d-1}.

In this case for (42) and (43) we have

$$I=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}^*}(\omega)J(\omega)\,d\omega}{\left(\sum_{k=1}^d\widetilde{t}_k^{2\nu}(\omega)\right)^{\widehat{q}^*(1-\widehat{\gamma})/2}},\tag{50}$$

$$I'_j=\int_{\Pi^{d-1}}\frac{\widetilde{\psi}^{\widehat{q}^*}(\omega)\widetilde{t}_j^{2\nu}(\omega)J(\omega)\,d\omega}{\left(\sum_{k=1}^d\widetilde{t}_k^{2\nu}(\omega)\right)^{\widehat{q}^*(1-\widehat{\gamma})/2+1}},\ j=1,\dots,d.$$

Similar to how it was done for weights (36) we prove that $I<\infty$ and $I'_1=\ldots=I'_d$. Thus, from Theorem 6 we obtain

Corollary 7. *Let $2<p\le\infty$ and $\nu>\eta\ge 0$. Then*

$$E_{p2}(\Lambda,(D^{\nu e_1},\dots,D^{\nu e_d}))=\frac{1}{(2\pi)^{d\widehat{\gamma}/2}}C_p(\nu,\eta)I^{1/\widehat{q}^*}\delta^{\widehat{\gamma}},$$

where I is defined by (50)*. The method*

$$\widehat{m}(y)(t)=F^{-1}\left(\left(1-\beta\frac{\sum_{j=1}^d|t_j|^{2\nu}}{|\psi(t)|^2}\right)_+\psi(t)y(t)\right),$$

where

$$\beta = \frac{1-\widehat{\gamma}}{d(2\pi)^{d\widehat{\gamma}}} C_p^2(\nu,\eta) \left(\delta I^{1/2-1/p}\right)^{2\widehat{\gamma}},$$

is optimal.

The sharp inequality

$$\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{d\widehat{\gamma}/2}} C_p(\nu,\eta) I^{1/\widehat{q}^*} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\widehat{\gamma}} \left(\max_{1\le j\le d} \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}\right)^{1-\widehat{\gamma}}$$

holds.

As functions $\psi(\cdot)$ defining the operator Λ we can consider the functions

$$\psi_\theta(\xi) = (|\xi_1|^\theta + \ldots + |\xi_d|^\theta)^{2/\theta}, \quad \theta > 0.$$

The corresponding operator is denoted by Λ_θ. In particular, $\Lambda_2 = -\Delta$, where Δ is the Laplace operator. We denote by $\Lambda_\theta^{\eta/2}$ the operator Λ which is defined by $\psi(\cdot) = \psi_\theta^{\eta/2}(\cdot)$.

Now we consider the case when $p = 2$.

Theorem 7. *Let $\nu > \eta > 0$, $\nu \ge 1$, and $0 < \theta \le 2\nu$. Then*

$$E_{22}(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \ldots, D^{\nu e_d})) = d^{\eta/\theta} \left(\frac{\delta}{(2\pi)^{d/2}}\right)^{1-\eta/\nu}, \tag{51}$$

and all methods

$$\widehat{m}(y)(t) = F^{-1}\left(a(t)\psi_\theta^{\eta/2}(t) y(t)\right), \tag{52}$$

where $a(\cdot)$ are measurable functions satisfying the condition

$$\psi_\theta^\eta(\xi)\left(\frac{|1-a(\xi)|^2}{\lambda_2 \sum_{j=1}^d |\xi_j|^{2\nu}} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1}\right) \le 1, \tag{53}$$

in which

$$\lambda_1 = \frac{d^{2\eta/\theta}}{(2\pi)^d}\left(1 - \frac{\eta}{\nu}\right)\left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu}, \quad \lambda_2 = \frac{\eta}{\nu} d^{2\eta/\theta - 1}\left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu - 1},$$

are optimal.

The sharp inequality

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \le \frac{d^{\eta/\theta}}{(2\pi)^{d(1-\eta/\nu)/2}} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{\eta/\nu} \left(\max_{1\le j\le d} \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}\right)^{1-\eta/\nu} \tag{54}$$

holds.

Proof. It follows by Lemma 1 that

$$E_{22}(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \ldots, D^{\nu e_d})) \ge \sup_{\substack{x(\cdot) \in W_2^{(D^{\nu e_1}, \ldots, D^{\nu e_d})} \\ \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta}} \|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}. \tag{55}$$

Given $0 < \varepsilon < (2\pi)^{d/(2\nu)} \delta^{-1/\nu}$, we set

$$\widehat{\xi}_\varepsilon = \left(\frac{(2\pi)^d}{\delta^2}\right)^{\frac{1}{2\nu}} (1, \ldots, 1) - (\varepsilon, \ldots, \varepsilon), \quad B_\varepsilon = \{\xi \in \mathbb{R}^d : |\xi - \widehat{\xi}_\varepsilon| < \varepsilon\}.$$

Consider a function $x_\varepsilon(\cdot)$ such that

$$Fx_\varepsilon(\xi)=\begin{cases}\dfrac{\delta}{\sqrt{\operatorname{mes} B_\varepsilon}}, & \xi\in B_\varepsilon,\\ 0, & \xi\notin B_\varepsilon.\end{cases} \tag{56}$$

Then $\|Fx_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\delta^2$ and

$$\|D^{\nu e_j}x_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{\delta^2}{(2\pi)^d\operatorname{mes} B_\varepsilon}\int_{B_\varepsilon}|\xi_j|^{2\nu}\,d\xi\le 1,\ j=1,\dots,d.$$

By virtue of (55) we have

$$E^2_{22}(\Lambda^{\eta/2}_\theta,(D^{\nu e_1},\dots,D^{\nu e_d}))\ge\|\Lambda^{\eta/2}_\theta x_\varepsilon(\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$=\frac{\delta^2}{(2\pi)^d\operatorname{mes} B_\varepsilon}\int_{B_\varepsilon}\psi^\eta_\theta(\xi)\,d\xi=\frac{\delta^2}{(2\pi)^d}\psi^\eta_\theta(\widetilde{\xi}_\varepsilon),\quad \widetilde{\xi}_\varepsilon\in B_\varepsilon.$$

Letting $\varepsilon\to0$ we obtain the estimate

$$E^2_{22}(\Lambda^{\eta/2}_\theta,(D^{\nu e_1},\dots,D^{\nu e_d}))\ge d^{2\eta/\theta}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-\eta/\nu}. \tag{57}$$

We will find optimal methods among methods (52). Passing to the Fourier transform we have

$$\|\Lambda^{\eta/2}_\theta x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi^\eta_\theta(\xi)\,|Fx(\xi)-a(\xi)y(\xi)|^2\,d\xi.$$

We set $z(\cdot)=Fx(\cdot)-y(\cdot)$ and note that

$$\int_{\mathbb{R}^d}|z(\xi)|^2\,d\xi\le\delta^2,\quad \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi_j|^{2\nu}|Fx(\xi)|^2\,d\xi\le1,\ j=1,\dots,d.$$

Then

$$\|\Lambda^{\eta/2}_\theta x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi^\eta_\theta(\xi)\,|(1-a(\xi))\,Fx(\xi)+a(\xi)z(\xi)|^2\,d\xi.$$

We write the integrand as

$$\psi^\eta_\theta(\xi)\left|\frac{(1-a(\xi))\sqrt{\lambda_2}\left(\sum_{j=1}^d|\xi_j|^{2\nu}\right)^{1/2}Fx(\xi)}{\sqrt{\lambda_2}\left(\sum_{j=1}^d|\xi_j|^{2\nu}\right)^{1/2}}+\frac{a(\xi)}{(2\pi)^{d/2}\sqrt{\lambda_1}}(2\pi)^{d/2}\sqrt{\lambda_1}z(\xi)\right|^2.$$

Applying the Cauchy-Bunyakovskii-Schwarz inequality we obtain the estimate

$$\|\Lambda^{\eta/2}_\theta x(\cdot)-\widehat{m}(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$\le\operatorname*{vraisup}_{\xi\in\mathbb{R}^d}S(\xi)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}|Fx(\xi)|^2+(2\pi)^d\lambda_1|z(\xi)|^2\right)d\xi,$$

where

$$S(\xi)=\psi^\eta_\theta(\xi)\left(\frac{|1-a(\xi)|^2}{\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}}+\frac{|a(\xi)|^2}{(2\pi)^d\lambda_1}\right).$$

If we assume that $S(\xi)\le 1$ for almost all ξ, then taking into account (57), we get

$$e_{22}^2(\Lambda_\theta^{\eta/2},(D^{\nu e_1},\dots,D^{\nu e_d}),\widehat{m})\le\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}|Fx(\xi)|^2+(2\pi)^d\lambda_1|z(\xi)|^2\right)d\xi$$
$$\le\lambda_2 d+\lambda_1\delta^2=d^{2\eta/\theta}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-\eta/\nu}\le E_{22}^2(\Lambda_\theta^{\eta/2},(D^{\nu e_1},\dots,D^{\nu e_d})).$$

This proves (51) and shows that the methods under consideration are optimal.

It remains to verify that the set of functions $a(\cdot)$ satisfying (53) is nonempty. Put

$$a(\xi)=\frac{(2\pi)^d\lambda_1}{(2\pi)^d\lambda_1+\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}}.$$

Then

$$S(\xi)=\frac{\psi_\theta^\eta(\xi)}{(2\pi)^d\lambda_1+\lambda_2\sum_{j=1}^d|\xi_j|^{2\nu}}.$$

Since $\theta\le 2\nu$ by Hölder's inequality

$$\sum_{j=1}^d|\xi_j|^\theta\le\left(\sum_{j=1}^d|\xi_j|^{2\nu}\right)^{\theta/(2\nu)}d^{1-\theta/(2\nu)}.$$

Putting $\rho=(|\xi_1|^\theta+\ldots+|\xi_d|^\theta)^{1/\theta}$, we obtain

$$\sum_{j=1}^d|\xi_j|^{2\nu}\ge\rho^{2\nu}d^{1-2\nu/\theta}.$$

Thus,

$$S(\xi)\le\frac{\rho^{2\eta}}{(2\pi)^d\lambda_1+\lambda_2\rho^{2\nu}d^{1-2\nu/\theta}}.$$

It is easily checked that the function $f(\rho)=(2\pi)^d\lambda_1+\lambda_2\rho^{2\nu}d^{1-2\nu/\theta}-\rho^{2\eta}$ reaches a minimum on $[0,+\infty)$ at

$$\rho_0=d^{1/\theta}\left(\frac{(2\pi)^d}{\delta^2}\right)^{1/(2\nu)}.$$

Moreover, $f(\rho_0)=0$. Consequently, $f(\rho)\ge 0$ for all $\rho\ge 0$. Hence $S(\xi)\le 1$ for all ξ.

Inequality (54) is proved by the analogy with the proof of Corollary 1. □

2. **Recovery in the metric $L_\infty(\mathbb{R}^d)$.** Put

$$\gamma_1=\frac{\nu-\eta-d/2}{\nu+d(1/2-1/p)},\quad q_1=\frac{1}{1/2+\gamma_1(1/2-1/p)},$$
$$\widetilde{C}_p(\nu,\eta)=\gamma_1^{-\frac{\gamma_1}{p}}\left(\frac{1-\gamma_1}{n}\right)^{-\frac{1-\gamma_1}{2}}\left(\frac{B\left(q_1\gamma_1/p+1,q_1(1-\gamma_1)/2\right)}{2|\nu-\eta-d/2|}\right)^{1/q_1}.$$

For $1<p<\infty$ we define $k(\cdot)$ by the equality

$$\frac{k(t)}{(1-k(t))^{p-1}}=(2\pi)^d\frac{|\psi(t)|^{p-2}}{s_2^{p-1}(t)}.$$

We set

$$k(t)=\begin{cases}\min\left\{1,(2\pi)^d|\psi(t)|^{-1}\right\}, & p=1,\\ \left(1-s_2(t)|\psi(t)|^{-1}\right)_+, & p=\infty.\end{cases}$$

Theorem 8. *Let* $1 \le p \le \infty$, $\gamma_1 \in (0,1)$. *Assume that*

$$I = \int_{\Pi^{d-1}} \frac{\widetilde{\psi}^{q_1}(\omega)}{\widetilde{s}_2^{q_1(1-\gamma_1)/2}(\omega)} J(\omega)\, d\omega < \infty$$

and $I_1' = \ldots = I_n'$, *where*

$$I_j' = \int_{\Pi^{d-1}} \frac{\widetilde{\psi}^{q_1}(\omega)\widetilde{\varphi}_j^2(\omega)}{\widetilde{s}_2^{q_1(1-\gamma_1)/2+1}(\omega)} J(\omega)\, d\omega, \ j = 1, \ldots, n.$$

Then

$$E_{p\infty}(\Lambda, \mathcal{D}) = \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_p(\nu, \eta) I^{1/q_1} \delta^{\gamma_1}.$$

The method

$$\widehat{m}(y)(t) = F^{-1}\left(k\left(\xi_1^{\frac{1}{n+d(1/2-1/p)}} t\right) \psi(t) y(t)\right),$$

where

$$\xi_1 = \delta \gamma_1^{-\frac{q_1}{2p}} \left(\frac{(1-\gamma_1)\widetilde{C}_p(\nu,\eta) I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}}\right)^{q_1(1/2-1/p)},$$

is optimal.

The sharp inequality

(58)
$$\|\Lambda x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_p(\nu,\eta) I^{1/q_1} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\gamma_1} \left(\max_{1 \le j \le n} \|D_j x(\cdot)\|_{L_2(\mathbb{R}^d)}\right)^{1-\gamma_1}$$

holds.

Proof. Using an estimate similar to (48) we have

$$E_{p\infty}(\Lambda, \mathcal{D}) \ge \sup_{\substack{x(\cdot) \in W_p^{\mathcal{D}} \\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta}} \|\Lambda x(\cdot)\|_{L_\infty(\mathbb{R}^d)}.$$

Assume that $x(\cdot) \in W_p^{\mathcal{D}}$ and $\|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta$. If $\widehat{x}(\cdot)$ is such that $F\widehat{x}(\xi) = \varepsilon(\xi) e^{-i\langle t,\xi\rangle} Fx(\xi)$, where

$$\varepsilon(\xi) = \begin{cases} \dfrac{\overline{\psi(\xi)Fx(\xi)}}{|\psi(\xi)Fx(\xi)|}, & \psi(\xi)Fx(\xi) \ne 0, \\ 0, & \psi(\xi)Fx(\xi) = 0, \end{cases}$$

then we obtain $\widehat{x}(\cdot) \in W_p^{\mathcal{D}}$, $\|F\widehat{x}(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta$ and

$$\left|\int_{\mathbb{R}^d} \psi(\xi) F\widehat{x}(\xi) e^{i\langle t,\xi\rangle}\, d\xi\right| = \int_{\mathbb{R}^d} |\psi(\xi)Fx(\xi)|\, d\xi.$$

Hence

(59)
$$E_{p\infty}(\Lambda, \mathcal{D}) \ge \frac{1}{(2\pi)^d} \sup_{\substack{x(\cdot) \in W_p^{\mathcal{D}} \\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta}} \int_{\mathbb{R}^d} |\psi(\xi)Fx(\xi)|\, d\xi.$$

Let $1 \le p < \infty$. It follows from (20) that

$$E_{p\infty}(\Lambda, \mathcal{D}) \ge E(p,1,2),$$

where, in the problem of the evaluation of $E(p,1,2)$, the functions $\varphi_j(\cdot)$ should be replaced by the function $(2\pi)^{-d/2}\varphi_j(\cdot)$, and the function $\psi(\cdot)$ by $(2\pi)^{-d}\psi(\cdot)$. From Theorem 5 we obtain

$$E_{p\infty}(\Lambda,\mathcal{D})\ge\frac{1}{(2\pi)^{d(1+\gamma_1)/2}}K\delta^{\gamma_1},$$

where

$$K=\gamma_1^{-\frac{\gamma_1}{p}}\left(\frac{1-\gamma_1}{n}\right)^{-\frac{1-\gamma_1}{2}}\left(\frac{B\left(q_1\gamma_1/p,q_1(1-\gamma_1)/2\right)I}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)}\right)^{1/q_1}.$$

From the properties of the beta-function

$$\begin{aligned}\frac{B\left(q_1\gamma_1/p,q_1(1-\gamma_1)/2\right)}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)}&=\frac{B\left(q_1\gamma_1/p+1,q_1(1-\gamma_1)/2\right)\left(q_1\gamma_1/p+q_1(1-\gamma_1)/2\right)}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)q_1\gamma_1/p}\\&=\frac{B\left(q_1\gamma_1/p+1,q_1(1-\gamma_1)/2\right)}{2|\nu-\eta-d/2|}.\end{aligned}$$

Thus,

$$E_{p\infty}(\Lambda,\mathcal{D})\ge\frac{1}{(2\pi)^{d(1+\gamma_1)/2}}\widetilde{C}_p(\nu,\eta)I^{1/q_1}\delta^{\gamma_1}.$$

Moreover, it follows from the same Theorem 5 that

$$\int_{\mathbb{R}^d}\left|\frac{1}{(2\pi)^d}\psi(\xi)F(\xi)-m(y)(\xi)\right|d\xi\le E(p,1,2),$$

where

$$m(y)(t)=\frac{1}{(2\pi)^d}k\left(\xi_1^{\frac{1}{\nu+d(1/2-1/p)}}t\right)\psi(t)y(t),$$

and

$$\begin{aligned}\xi_1&=\frac{\delta}{\gamma_1^{1/p}}\left(\frac{1-\gamma_1}{n}\right)^{1/2}\left(\frac{B\left(q_1\gamma_1/p,q_1(1-\gamma_1)/2\right)I(2\pi)^{-dq_1(1+\gamma_1)/2}}{|\nu+d(1/2-1/p)|(2\gamma_1+(1-\gamma_1)p)}\right)^{1/2-1/p}\\&=\delta\gamma_1^{-\frac{q_1}{2p}}\left(\frac{(1-\gamma_1)\widetilde{C}_p(\nu,\eta)I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}}\right)^{q_1(1/2-1/p)}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left|\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi(\xi)F(\xi)e^{i\langle t,\xi\rangle}\,d\xi-\int_{\mathbb{R}^d}m(y)(\xi)e^{i\langle t,\xi\rangle}\,d\xi\right|\\&\qquad\le\int_{\mathbb{R}^d}\left|\frac{1}{(2\pi)^d}\psi(\xi)F(\xi)-m(y)(\xi)\right|d\xi\le E(p,1,2)\le E_{p\infty}(\Lambda,\mathcal{D}).\end{aligned}$$

It follows that the method $\widehat{m}(y)(\cdot)$ is optimal, and the error of optimal recovery coincides with $E(p,1,2)$.

Now we consider the case when $p=\infty$. Put

$$s(\xi)=\begin{cases}\dfrac{\psi(\xi)}{|\psi(\xi)|}, & \psi(\xi)\ne 0,\\ 1, & \psi(\xi)=0.\end{cases}$$

Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi)=\begin{cases}\delta\overline{s(\xi)}, & |\psi(\xi)|\ge\lambda s_2(\xi),\\ \dfrac{\delta\overline{\psi(\xi)}}{\lambda s_2(\xi)}, & |\psi(\xi)|<\lambda s_2(\xi).\end{cases}$$

We choose $\lambda>0$ such that

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\varphi_j(\xi)|^2|F\widehat{x}(\xi)|^2\,d\xi=1,\quad j=1,\dots,n.$$

Now, to find λ we have the equation

$$\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|\ge\lambda s_2(\xi)}|\varphi_j(\xi)|^2\,d\xi+\frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int_{|\psi(\xi)|<\lambda s_2(\xi)}\frac{|\varphi_j(\xi)|^2|\psi(\xi)|^2}{s_2^2(\xi)}\,d\xi=1.$$

If $\nu>\eta+d/2$, then from the fact that $\gamma_1\in(0,1)$ it follows that $\eta>-d$. In this case it is easy to check that $2\nu>\eta$ and $2\nu+d>0$. Passing to the polar transformation we obtain

$$\frac{\delta^2}{(2\pi)^d}\int_{\Pi_{d-1}}\widetilde{\varphi}_j^2(\omega)J(\omega)\,d\omega\int_0^{\Phi_2(\omega)}\rho^{2\nu+d-1}\,d\rho$$
$$+\frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int_{\Pi_{d-1}}\frac{\widetilde{\varphi}_j^2(\omega)\widetilde{\psi}^2(\omega)}{\widetilde{s}_2^2(\omega)}J(\omega)\,d\omega\int_{\Phi_2(\omega)}^{+\infty}\rho^{-2\nu+2\eta+d-1}\,d\rho=1,$$

where

$$\Phi_2(\omega)=\left(\frac{\widetilde{\psi}(\omega)}{\lambda\widetilde{s}_2(\omega)}\right)^{\frac{1}{2\nu-\eta}}.$$

Thus,

$$\frac{\delta^2}{(2\pi)^d}\lambda^{-\frac{2\nu+d}{2\nu-\eta}}\frac{4\nu-2\eta}{(2\nu+d)(2\nu-2\eta-d)}I_j=1.$$

If $\nu<\eta+d/2$, then it follows from $\gamma_1\in(0,1)$ that $\eta<-d$, $2\nu<\eta$, and $2\nu+d<0$. Passing to the polar transformation we obtain

$$\frac{\delta^2}{(2\pi)^d}\int_{\Pi_{d-1}}\widetilde{\varphi}_j^2(\omega)J(\omega)\,d\omega\int_{\Phi_2(\omega)}^{+\infty}\rho^{2\nu+d-1}\,d\rho$$
$$+\frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int_{\Pi_{d-1}}\frac{\widetilde{\varphi}_j^2(\omega)\widetilde{\psi}^2(\omega)}{\widetilde{s}_2^2(\omega)}J(\omega)\,d\omega\int_0^{\Phi_2(\omega)}\rho^{-2\nu+2\eta+d-1}\,d\rho=1,$$

For this case we have

$$\frac{\delta^2}{(2\pi)^d}\lambda^{-\frac{2\nu+d}{2\nu-\eta}}\frac{2\eta-4\nu}{(2\nu+d)(2\nu-2\eta-d)}I_j=1.$$

Combining both of these cases and taking into account that $I_j=I/n$, $j=1,\dots,n$, we get

$$\lambda=\left(\frac{2\delta^2|2\nu-\eta|I}{(2\pi)^d n(2\nu+d)(2\nu-2\eta-d)}\right)^{\frac{2\nu-\eta}{2\nu+d}}.$$

It follows by (59) that

$$E_{\infty\infty}(\Lambda,\mathcal{D})\ge\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\psi(\xi)F\widehat{x}(\xi)|\,d\xi=\frac{\delta}{(2\pi)^d}\int_{|\psi(\xi)|\ge\lambda s_2(\xi)}|\psi(\xi)|\,d\xi$$
$$+\frac{\delta}{\lambda(2\pi)^d}\int_{|\psi(\xi)|<\lambda s_2(\xi)}\frac{|\psi(\xi)|^2}{s_2(\xi)}\,d\xi.$$

Using calculations similar to those that were above, we obtain

$$E_{\infty\infty}(\Lambda,\mathcal{D})\geq\frac{\delta|2\nu-\eta|\lambda^{-\frac{\eta+d}{2\nu-\eta}}I}{(2\pi)^d(\eta+d)(2\nu-2\eta-d)}=E_0,$$

where

$$E_0=\frac{(n|\nu+d/2|)^{\frac{\eta+d}{2\nu+d}}}{\eta+d}\left(\frac{(2\nu-\eta)I}{(2\pi)^d(2\nu-2\eta-d)}\right)^{\frac{2\nu-\eta}{2\nu+d}}\delta^{\frac{2\nu-2\eta-d}{2\nu+d}}.$$

We prove that for all $x(\cdot)\in X_\infty$ the equality

$$\Lambda x(t)=\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi+\frac{\lambda}{\delta(2\pi)^d}\int_{\mathbb{R}^d}s_2(\xi)Fx(\xi)\overline{F\widehat{x}(\xi)}e^{i\langle t,\xi\rangle}\,d\xi. \tag{60}$$

holds. Indeed,

$$\begin{aligned}&\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\\&\qquad+\frac{\lambda}{\delta(2\pi)^d}\int_{\mathbb{R}^d}s_2(\xi)Fx(\xi)\overline{F\widehat{x}(\xi)}e^{i\langle t,\xi\rangle}\,d\xi\\&=\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}((\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\\&+\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}\lambda s(\xi)s_2(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi+\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|<\lambda s_2(\xi)}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\\&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi=\Lambda x(t).\end{aligned}$$

We estimate the error of the method

$$m(y)(t)=\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,y(\xi)e^{i\langle t,\xi\rangle}\,d\xi.$$

We have

$$\begin{aligned}|\Lambda x(t)-m(y)(t)|\leq\Bigg|&\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\\&-\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}(\psi(\xi)-\lambda s(\xi)s_2(\xi))\,Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\Bigg|\\&+\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}|\psi(\xi)-\lambda s(\xi)s_2(\xi)|\,|Fx(\xi)-y(\xi)|\,d\xi.\end{aligned}$$

If $x(\cdot)$ such that

$$\|Fx(\cdot)-y(\cdot)\|_{L_\infty(\mathbb{R}^d)}\leq\delta,\quad\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\varphi_j(\xi)|^2|Fx(\xi)|^2\,d\xi\leq1,\ j=1,\dots,n,$$

then, taking into account (60), we obtain

$$|\Lambda x(t)-m(y)(t)|\leq\frac{\lambda}{\delta(2\pi)^d}\int_{\mathbb{R}^d}s_2(\xi)|Fx(\xi)||F\widehat{x}(\xi)|\,d\xi+\mu\leq\frac{n\lambda}{\delta}+\mu,$$

where

$$\mu=\frac{\delta}{(2\pi)^d}\int_{|\psi(\xi)|\geq\lambda s_2(\xi)}(|\psi(\xi)|-\lambda s_2(\xi))\,d\xi.$$

Passing to the polar transformation we find

$$\frac{\delta}{(2\pi)^d}\int_{|\psi(\xi)|\ge\lambda s_2(\xi)}|\psi(\xi)|\,d\xi=\frac{\delta\lambda^{-\frac{\eta+d}{2\nu-\eta}}}{(2\pi)^d|\eta+d|}I,$$
$$\frac{\delta\lambda}{(2\pi)^d}\int_{|\psi(\xi)|\ge\lambda s_2(\xi)}s_2(\xi)\,d\xi=\frac{\delta\lambda^{-\frac{\eta+d}{2\nu-\eta}}}{(2\pi)^d|2\nu+d|}I.$$

Hence

$$\mu=\frac{\delta\lambda^{-\frac{\eta+d}{2\nu-\eta}}|2\nu-\eta|}{(2\pi)^d(\eta+d)(2\nu+d)}I.$$

It is easily checked that $n\lambda/\delta+\mu=E_0$, and therefore

$$e_{\infty\infty}(\Lambda,\mathcal{D},m)\le E_0\le E_{\infty\infty}(\Lambda,\mathcal{D}).$$

It follows that $m(y)(\cdot)$ is an optimal method, and the error of optimal recovery is E_0. It is easily checked that for $p=\infty$

$$\frac{1}{(2\pi)^{d(1+\gamma_1)/2}}\widetilde{C}_\infty(\nu,\eta)I^{1/q_1}\delta^{\gamma_1}=E_0.$$

We evaluate ξ_1 for $p=\infty$. We have

$$\xi_1=\delta\left(\frac{(1-\gamma_1)\widetilde{C}_\infty(\nu,\eta)I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}}\right)^{q_1/2}=\lambda^{\frac{\nu+d/2}{2\nu-\eta}}. \tag{61}$$

The method $m(y)(\cdot)$ can be written as

$$m(y)(t)=F^{-1}\left(\left(1-\lambda\frac{s_2(\xi))}{|\psi(t)|}\right)_+\psi(t)y(t)\right).$$

In view of (61) we have

$$m(y)(t)=F^{-1}\left(k\left(\xi_1^{\frac{1}{n+d/2}}t\right)\psi(t)y(t)\right)=\widehat{m}(y)(t).$$

Inequality (58) is proved by the analogy with the proof of Corollary 1. □

It is not difficult to formulate a corollary from Theorem 8 analogous to Corollary 7 for the same Λ and $\mathcal{D}=(D^{\nu e_1},\dots,D^{\nu e_d})$.

References

[1] F.I. Andrianov, Multidimensional analogues of Carlson's inequality and its generalizations, Izv. Vyssh. Uchebn. Zaved. Mat. 1 (56) (1960) 3–7 (in Russian).
[2] S. Barza, V. Burenkov, J. Pečarić, L.-E. Persson, Sharp multidimentional multiplicative inequalities for weighted L_p spaces with homogeneous weights, Math. Ineq. Appl. 1 (1998) 53–67.
[3] F. Carlson, Une inégalité, Ark. Mat. Astr. Fysik 25B (1934) 1–5.
[4] V.I. Levin, Sharp constants in inequalities of Carlson type, Dokl. Akad. Nauk. SSSR 59 (1948) 635–638 (in Russian).
[5] C. A. Micchelli, T. J. Rivlin, A survey of optimal recovery, in Optimal Estimation in Approximation Theory, C. A. Micchelli and T. J. Rivlin, eds., Plenum Press, New York, 1977, pp. 1–54.
[6] K. Yu. Osipenko, Optimal Recovery of Analytic Functions, Nova Science Publ., Huntington, New York, 2000.
[7] K.Yu. Osipenko, Optimal recovery of linear operators in non-Euclidean metrics, Sb. Math. 205 (10) (2014) 1442–1472.
[8] K.Yu. Osipenko, Optimal recovery of operators and multidimensional Carlson type inequalities, J. Complexity 32 (1) (2016) 53–73.

[9] K. Yu. Osipenko, Inequalities for derivatives with the Fourier transform, Appl. Comp. Harm. Anal., 53 (2021) 132–150.
[10] K. Yu. Osipenko, Optimal recovery in weighted spaces with homogeneous weights, Sbornic: Mathematics, 213 (2022) 385-411.
[11] K. Yu. Osipenko, Introduction to Optimal Recovery Theory, Lan Publishing House, St. Petersburg, 2022 (in Russian).
[12] L. Plaskota, Noisy Information and Computational Complexity, Cambridge University Press, Cambridge, 1996.
[13] J. F. Traub, H. Woźniakowski, A General Theory of Optimal Algorithms, Academic Press, New York, 1980.

Moscow State University, Moscow and Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow
E-mail address: kosipenko@yahoo.com